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On the space of closed subsets of a uniform space

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1. The family X of closed nonempty subsets of a metric seace E can be made into a metric space. When E is complete, so is X. If E is tatally bounded then so is X [Ⅲ. §15, §29]. If E compact then X is compact [Ⅱ. §28].

In a similar way, the family X(Y) of the nonempty closed subsets of E(F) relative to the Hausdorff uniform topology for E(F) can be made into a Hausdorff uniformity for X. Robertson[Ⅳ] proved that the space of X consisting all the nonempty compact subsets in E is always complete provided that E is complete in the Hausdorff uniform space.

In this Hausdorff uniform space, the first result of this paper shows that if E is tatally bounded then X is totally bounded. The second result is that if F is a closed subspace of E then Y is a closed subspace of X. And the third result is that X is compact if E is compact.

2. *Lemma* 1

A uniform space is compact iff it is tatally bounded and complete [Ⅰ. 6.32]

Suppose that E be a uniform Hausdorff space and u be a base for the uniformity for E and that X be the family of all closed subsets of E relative to the uniform topolcgy for E. Let v be the family of all sets of the form

$$W_U=\{(A.B):\ A\subset U[B],\ B\subset U[A] \text{ in X for eash U in } u\}$$

then each member of v contains the diagonal. Since each member of v is symmetric if $W_U \in v$ then the inverse of W_U contains a member of v. Let V be a member of u such that $VoV\subset U$ then $W_VoW_V\subset W_U$. Therefor if $W_U\in v$ then there exists W_V such that $W_VoW_V\subset W_U$ for some W_V in v. If $W\subset U\cap V$ then $W_W\subset W_U\cap W_V$, therefor the intersection of two member of v contains a member of v. And if $\cap\{U:U\in U\}$ is the diagonal then $\cap\{W_U:W_U\in v\}$ is also diagonal. Consequently X is a uniform Hausdorff space.

Theorem 1

If uniform Hausdorff space(E.u) is totally bounded then the uniform Hausdorff space (X.v) of closed nonempty subsets of E is also tatally bounded.

(*Proof*) Since E is totally bounded, for each U in u there is a finite subset F of E such that $E\subset U[F]$. Let H be the family of all subsets of F then H is a finite subfamily of X. Let A be a closed subset of E and U be a symmetric member of u then for each element x of A there is a member x_n of F such that $x\in U[x_n]$ and $x_n\in U[x]$ and hence $x_n\in U[A]$.

Let $B_i=\{x_n:\ x_n\in U[A]\ \ x_n\in F\}$ then B_i ie a member of H and B_i is nonempty. It follows that $B_i\subset U[A]$ and if $x\in A$ then

$$x\in U[x_n]\subset U[B_i] \text{ and hence } A\subset U[B_i].$$

This fact implies that $(A_i,\ B_i)\in\ W_U$ and $A\subset$

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$W_U(H)$ Therefor $X = W_U(H)$ is totally bounded.

Theorem 2

> If F is a closed subspace of E then Y is a closed subspace of X

(*proof*) If $B \in \bar{Y}$ then each neighborhood of B intersection with Y is nonempty, that is for all U in u $W_U(B) \cap Y \neq 0$. Hence there is an element A of Y which is closed subset of F and such that $A \in W_U(B)$. It follows that $B \subset U(A)$ and hence $B \subset U(F)$ for all U in u. Therefor $B \subset \cap \{U(F): U \in u\} = F$. Since B is a closed subset of F, B belongs to Y. consequently $\bar{Y} = Y$, that is, Y is closed subspace of X.

Theorem 3 (Robertson [Ⅳ])

> Let E be a complete uniform Hausdorff space X be the space of closed nonempty subsets of E under the Hausdorff uniform structure then nonempty compact subsets of X form a complete subspace of X.

Theorem 4:

> If Hausdorff uniform space is compact then X is compact

(*proof*) Since closed subset of compact space is compact if E is compact then the family of all nonempty compact subsets of E is just X. Therefor by theorem 3, X is complete. And according to Lemm 1, if E is compact then E is totally bounded. Hence by Lemma l, X is complete and totally bounded iff X is Compact.

References

Ⅰ. J.L. Kelley. Geneeal Topology. New Yoek Van Nostrand 1957.

Ⅱ. F. Hausdorff. Set theory 3ed. New York. CHelsea. 1962.

Ⅲ. C. Kuratowski Topologies, Volume 1, 3ed. Monogratje Matometyozne Warsaw 1952.

Ⅳ. A.P. Robertson and Wendy Robertson. "On the space of subsets of a uniform space" Proc, Amer. Math. Soc. Volume 12. 1961, p.321—p. 326.