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ON NEGATIVE MOMENTS OF POSITIVE RANDOM VARIABLES

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§ 0. INTRODUCTION

We investigate the problem of finding the expected value of functions of a random variable X, of the form

$$f(x)=(x+A)^{-n},$$

where $X+A>0$, and n is a non-negative integer.

It had been investigated the problem with inductive definition by Chao and Strawderman [1], but we obtain more simply results an easy way without inductive definition.

We develop the technique in Section 1 and apply it to finding

$$E\left[\frac{1}{X+A}\right]$$

for the binomial and poisson distributions in Section 2.

Negative moments are useful in applications in several contents, notably in life testing problems, and survey sampling problems where ratio estimates are used. See [2, 3, 4, 8] for some applications.

The results in the literature seem to have been confined primarily to the case of the truncated Poisson and binomial distributions [5, 6, 7].

§ 1. THE RESULTS

Let X be a random variable defined on a probability space and suppose $X+A<\delta<0$.

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Define the probability generating function of $(X+A)^k-1$ as

$$g_k(t)=E[t^{(X+A)^k-1}] \qquad (2.1)$$
$$0\leq t\leq 1$$

Clearly (2.1) exist under the assumption on X. We have the following result.

Theorem. For $0\leq t\leq 1$,

$$E\left[\left(\frac{1}{X+A}\right)^k\right]=\int_0^1 g_k(t)\,dt$$

Proof: Let X be a discrete random variable with probability function P(x).

$$\begin{aligned}\int_0^1 g_k(t)dt&=\int_0^1 E[t^{X+A)^k-1}]dt\\&=\int_0^1\sum_x t^{(X+A)^k-1}\cdot p(x)dt\\&=\sum_x\int_0^1 t^{(X+A)^k-1}\,dt\cdot p(x)\\&=\sum_x\left[\frac{t^{(X+A)^k}}{(X+A)}\right]_0^1 p(x)\\&=\sum_x\frac{1}{(X+A)^n}\cdot p(x)\\&=E\left[\left(\frac{1}{X+A}\right)^k\right]\end{aligned}$$

Let X be a continuous random variable with probabilty density function $\varphi(x)$.

$$\begin{aligned}\int_0^1 g_k(t)dt&=\int_0^1 E[t^{(X+A)^k-1}]dt\\&=\int_0^1\left[\int_{-\infty}^{\infty}t^{(X+A)^k-1}\cdot\varphi(x)dx\right]dt\\&=\int_{-\infty}^{\infty}\left[\int_0^1 t^{(X+A)^k-1}dt\right]\varphi(x)dx\end{aligned}$$

$$=\int_{-\infty}^{\infty}\frac{1}{(X+A)^{k}}\cdot\varphi(x)\,dx$$
$$=E\left[\left(\frac{1}{X+A}\right)^{k}\right]\qquad\blacksquare$$

We have immediately, the Corollary

$$E\left[\frac{1}{X+A}\right]=\int_0^1 g_1(t)\,dt$$

When k is equal to 1 on the corollary of [1], it is correspond with our corollary.

The potential applicability of the corollary is immediately of the corollary is immediately evident. In Section 2 we apply it to the binomial and Poisson cases. Since we can find these applications in [1], it is sufficient to mention here a few simple examples.

§2. APPLICATIONS

2.1. Binomial Distribution

Let X be a binomially distributed random variable with parameters n and p. It is easy to show

$$g_1(t)=t^{A-1}(q+pt)^{n},$$

and using successive integrations by parts we are led to

$$E\left[\frac{1}{X+A}\right]=\int_0^1 g_1(t)\,dt$$
$$=\int_0^1 t^{A-1}(q+pt)^{n}dt$$
$$=q^{n}\left(\frac{q}{p}\right)^{A}\left[\sum_{k=1}^{A-1}(-1)^{k+1}\right.$$
$$\times\frac{(A-1)(A-2)\cdots(A-k+1)}{(n+1)(n+2)\cdots(n+k)}$$
$$\cdot\left(\frac{p}{q}\right)^{A-k}\cdot\left(1+\frac{p}{q}\right)^{n+k}$$
$$+(1-)^{A-1}\frac{(A-1)!}{(n+1)(n+2)\cdots(n+A-1)}$$
$$\left.\cdot\frac{1}{n+A}\cdot\left(\left(1+\frac{p}{q}\right)^{n+A}-1\right)\right],$$

where, by convention, $(A-1)(A-2)\cdots(A-k)=1$ if $k=o$; and where $A-r>0$,

In particular,

$$E\left(\frac{1}{X+1}\right)=\frac{1-q^{n+1}}{(n+1)p}$$

2.2. The Poisson Distribution

Let X be a random variable with the Poisson distribution wtih parameter λ.

Here

$$g_1(t)=t^{A-1}e^{-\lambda+\lambda t}$$
$$E\left[\frac{1}{X+A}\right]=\int_0^1 g_1(t)\,dt$$
$$=\int_0^1 t^{A-1}e^{-\lambda+\lambda t}dt$$
$$=e^{-\lambda}\int_0^1 t^{A-1}e^{-\lambda t}dt\qquad\text{for } A>O$$
$$=e^{-\lambda}\left[\frac{e^{\lambda}}{\lambda}-\frac{A-1}{\lambda}\cdot\int t^{A-2}e^{\lambda t}dt\right]$$
$$\text{for } A<1$$

We have

$$E\left[\frac{1}{X+A}\right]=\frac{1}{\lambda}\left[1-(A-1)E\left(\frac{1}{X+A-1}\right)\right]$$
$$\text{for } A>1\qquad(3.1)$$

when $A=1$ we have directdy

$$E\left[\frac{1}{X+1}\right]=\frac{e^{-\lambda}}{\lambda}(e^{\lambda}-1)$$
$$=\frac{1-e^{-\lambda}}{\lambda}\qquad(3.2)$$

(3.1) and (3.2) together allow an inductive calculation of $E\left[\frac{1}{X+A}\right]$ for any integer $A\geq 1$.

REFERENCES

[1] Chao, M.T. and Strawderman, W.E., "Negative Moments of Positive Random Variables," *Journal of the American Statistical Association*, 67 (June 1972), 429—31.

[2] Bartholomen, A.J., "A Problem in Life Testing," *Journal of the American Statistical Association*, 52 (September 1957), 350—5.

[3] Epstein, B. and Sobel, M., "Some Theorems Relevant to Life Testing," *The Annals of Mathematicul Statistics*, 25 (June 1954), 373—81

[4] Epstein, B. and Sobel, M., "Life Testing," *Journal of the American Statistical Association*, 48 (September. 1953), 486—502.

[5] Grab, E.L. and Savage, I.R., "Tables of the Expected Value of 1/X for Positive Bernoulli and Poisson Variables," *Journal of the American Statistical Association*, 49, (March 1954), 169—77.

[6] Stephan, F.F., "The Expected Value and Variance of the Reciprocal and other Negative Powers of a Positive Bernoullian Variate," *The Annals of Mathematical Statistics*, 16 (March 1945), 50—61.

[7] Tiku, M.L., "A Note on the Negative Moments of a Truncated Poisson Variate." *Journal of the American Statistical Association*, 59 (December 1964), 1220—4.

[8] Deming, W.E., *Some Theory of Sampling*, New York: John Wiley and Sons, Inc., 1950, 449—54.

〔要　約〕

本論文은 n이 陰이 아닌 整數로서 $X+A>0$ 일 때,

$$f(x)=(X+A)^{-n}$$

모양의 確率變數 X의 期待値를 구하는 問題를 研究한 것이다.

이 問題는 이미 CHAO, STRAWDERMAN에 依해서 研究된바 있으나〔1〕, 여기서는 더욱 簡單하게 單純한 結果를 얻었으며 n=1일 때는 兩結果는 一致한다.