AN EXTENSION ON GENERALIZED HYPERGEOMETRIC POLYNOMIALS

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Summary.

In this paper, the author has established the formulae for product of two generalized hypergeometric polynomials by defining the polynomial in the form

$$F_n(x)=x^{(\delta-1)n}{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),a_1,\cdots\cdots,a_p\\ b_1,\cdots\cdots,b_q\end{matrix};\lambda x^c\right],$$

where the symbol $\Delta(\delta,-n)$ represents the set of δ-parameters:

$$\frac{-n}{\delta},\ \frac{-n+1}{\delta},\cdots\cdots,\ \frac{-n+\delta-1}{\delta}$$

and δ, n are positive integers. A number of known as well as new results have been also obtained with proper choice of parameters.

1. Introduction.

The generalized hypergeometric polynomial [(5), eqn. (2.1), p. 79] has been defined by means of

$$(1.1)\qquad F_n(x)=x^{(\delta-1)n}{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),\ a_1,\cdots\cdots,a_p\\ b_1,\cdots\cdots,b_q\end{matrix};\ \mu x^c\right],$$

where δ and n are positive integers, the symbol $\Delta(\delta,-n)$ stands for the set of δ-parameters:

$$\frac{-n}{\delta},\ \frac{-n+1}{\delta},\cdots\cdots,\ \frac{-n+\delta-1}{\delta}.$$

The polynomial has arisen in the course of an attempt to unify and extend the study of most of the well-known sets of polynomials.

For brevity and ease in writing, we employ the contracted notation

$${}_pF_q(x)={}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\ \middle|\ x\right)=\sum_{r=0}^{\infty}\frac{(a_p)_r\ x^r}{(b_q)_r\ r!}.$$

Thus $(a_p)_r$ is to be interpreted as $\prod_{j=1}^{p}(a_j)_r$ and similarly for $(b_q)_r$.

This paper is concerned with some formulae involving the product of two generalized hypergeometric polynomials in series. The polynomial is in a more generalized form which yields many known and new results on specializing the parameters. Therefore the results obtained in this paper are of general character.

2. Product of generalized hypergeometric polynomials.

Considering the product of two generalized hypergeometric polynomials by expressing the generalized hypergeometric polynomial (1.1) in series, we obtain

$$(2.1)\quad F_n(x)F_m(y)=x^{(\delta-1)n}{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),a_p\\ b_q\end{matrix};\mu x^c\right]y^{(\gamma-1)m}$$

$$\times{}_{l+\gamma}F_k\left[\begin{matrix}\Delta(\gamma,-m),\rho_l\\ \sigma_k\end{matrix};\lambda y^d\right]$$

$$=x^{(\delta-1)n}y^{(\gamma-1)m}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_r(a_r)_r\mu^r x^{cr}}{r!(b_q)_r}$$

$$\times\frac{\prod_{i=0}^{\gamma-1}\left(\frac{-m+i}{\gamma}\right)_s(\rho_l)_s\lambda^s y^{ds}}{s!(\sigma_k)_s},$$

where δ,γ,m and n are positive integers.

Replacing r by $r-s$ and using the known relation

$$(\alpha)_{n-k}=\frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}\text{ for }0\leq k\leq n,$$

we have

$$(2.2)\quad x^{(\delta-1)n}{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),a_p\\ b_q\end{matrix};\mu x^c\right]y^{(\gamma-1)m}{}_{l+\gamma}F_k\left[\begin{matrix}\Delta(\gamma,-m),\ \rho_l\\ \sigma_k\end{matrix};\lambda y^d\right]$$

$$=x^{(\delta-1)n}y^{(\gamma-1)m}\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_r(a_p)_r\mu^r x^{cr}}{r!(b_q)_r}$$

$$\times{}_{l+q+\gamma+1}F_{p+\delta+k}\left[\begin{matrix}\Delta(\gamma,-m),1-b_q-r,-r, & \rho_l\\ \Delta(\delta,n+1-r\delta),\ 1-a_p-r, & \sigma_k\end{matrix};\frac{\lambda}{\mu}\frac{y^d}{x^c}(-1)^{p-q+\delta-1}\right].$$

Also we have

$$(2.3)\quad x^{(\delta-1)n}{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),a_p\\ b_q\end{matrix};\mu x^c\right]y^{(\gamma-1)m}{}_{l+\gamma}F_k\left[\begin{matrix}\Delta(\gamma,-m),\rho_l\\ \sigma_k\end{matrix};\lambda y^d\right]$$

$$=x^{(\delta-1)n}y^{(\gamma-1)m}\sum_{s=0}^{\infty}\frac{\prod_{i=0}^{\gamma-1}\left(\frac{-m+i}{\gamma}\right)_s(\rho_l)_s\lambda^s y^{ds}}{s!(\sigma_k)_s}$$

$$\times{}_{p+k+\delta+1}F_{l+q+\gamma}\left[\begin{matrix}\Delta(\delta,-n), & 1-\sigma_k-s,\ -s,\ a_p\\ \Delta(\gamma,m+1-s\gamma), & 1-\rho_l-s, \quad b_q\end{matrix};\frac{\mu}{\lambda}\ \frac{x^c}{y^d}(-1)^{l+\gamma-k-1}\right].$$

Particular cases of (2.2) with $\delta=\gamma=c=d=1$ and $y=x$:

(i) Setting $b_1=-n,\ \sigma_1=-m$, we obtain a known result [(4), eqn. (3.5), p. 395].

(ii) Substituting $p=q=l=k=2,\ a_1=a, b_2=b, b_1=-n, b_2=c, \rho_1=a', \rho_2=b', \sigma_1=-m, \sigma_2=c'$, we have a known result [(2), eqn. (14), p. 187].

Similarly with proper choice of parameters, we may obtain the other known results [(2), eqns. (12), (13) & (15), p. 187].

(iii) Taking $p=q=2, l=k=2,\ a_1=\alpha, a_2=\beta, b_1=-n, b_2=\alpha+\beta+\frac{1}{2}, \rho_1=\alpha,\ \rho_2=\beta, \sigma_1=-m, \sigma_2=\alpha+\beta+\frac{1}{2},\ \lambda=\mu=1$, we obtain

$$\text{(2. 4)}\quad \left[{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \alpha+\beta+\frac{1}{2}\end{matrix};x\right)\right]^2=\sum_{r=0}^{\infty}\frac{(\alpha)_r\,(\beta)_r x^r}{r!(\alpha+\beta+\frac{1}{2})_r}\,{}_4F_3\left(\begin{matrix}-r,\alpha,\beta,\frac{1}{2}-\alpha-\beta-r\\ 1-\alpha-r, 1-\beta-r, \alpha+\beta+\frac{1}{2}\end{matrix};\right)$$

Using the known result [(1), eqn. (2.4), p. 186]:

$${}_4F_3\left(\begin{matrix}-m,\alpha,\beta,\frac{1}{2}-\alpha-\beta--m\\ 1-\alpha-m,\ 1-\beta-m,\ \alpha+\beta+\frac{1}{2}\end{matrix};\right)=\frac{(2\alpha)_m(2\beta)_m(\alpha+\beta)_m}{(\alpha)_m(\beta)_m(2\alpha+2\beta)_m}$$

on the right hand side of (2.4), we obtain an identity due to T. Clausen [(2), eqn. (1), p. 185].

(iv) With $p=q=1,\ l=k=2,\ \lambda=\mu=1,\ a_1=c-a-b,\ b_1=-n,\ \rho_1=a,\ \rho_2=b,\ \sigma_1=-m, \sigma_2=c$ and using Saalschütz' theorem, we get a known result due to Euler [(6), eqn. (5), p. 60].

(v) Taking $p=q=2, l=k=2, \lambda=\mu=1, a_1=\rho_1=\alpha, a_2=\rho_2=\beta,\ b_1=-n, b_2=\alpha+\beta-\frac{1}{2}, \sigma_1=-m, \sigma_2=\alpha+\beta+\frac{1}{2}$, we have

$$\text{(2, 5)}\quad {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta-\frac{1}{2}\end{matrix};x\right)\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta+\frac{1}{2}\end{matrix};x\right)=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r\,x^r}{r!(\alpha+\beta-\frac{1}{2})_r}$$

$$\times {}_4F_3\left(\begin{matrix}-r, \alpha, \beta, \frac{3}{2}-\alpha-\beta-r \\ 1-\alpha-r, 1-\beta-r, \alpha+\beta+\frac{1}{2}\end{matrix} : \right).$$

With the help of the known result [(1), eqn. (3.3), p. 187] :

$${}_4F_3\left(\begin{matrix}-m, \alpha, \beta, \frac{3}{2}-\alpha-\beta-m \\ 1-\alpha-m, 1-\beta-m, \alpha+\beta+\frac{1}{2}\end{matrix} : \right)=\frac{(2\alpha)_m(2\beta)_m(\alpha+\beta)_m(\alpha+\beta-\frac{1}{2})_m}{(\alpha)_m(\beta)_m(\alpha+\beta+\frac{1}{2})_m(2\alpha+2\beta-1)_m},$$

on the right of (2.5), we obtain a known result [(2), eqn. (8), p. 186].

(vi) Substituting $p=q=l=k=2$, $\lambda=\mu=1, a_1=\alpha, a_2=\beta, b_1=-n$, $b_2=\alpha+\beta-\frac{1}{2}$, $\rho_1=\alpha-1, \rho_2=\beta$, $\sigma_1=-m$, $\sigma_2=\alpha+\beta-\frac{1}{2}$, we have

$$(2.6)\quad {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \alpha+\beta-\frac{1}{2}\end{matrix} : x\right){}_2F_1\left(\begin{matrix}\alpha-1, \beta \\ \alpha+\beta-\frac{1}{2}\end{matrix} : x\right)=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r\, x^r}{r!(\alpha+\beta-\frac{1}{2})_r}$$

$$\times {}_4F_3\left(\begin{matrix}-r, \alpha-1, \beta, \frac{3}{2}-\alpha-\beta-r \\ 1-\alpha-r, 1-\beta-r, \alpha+\beta-\frac{1}{2}\end{matrix} : \right).$$

With the application of the known result [(1), eqn (3.4), p. 187].

$${}_4F_3\left(\begin{matrix}-m, \alpha, \beta-1, \frac{3}{2}-\alpha-\beta-m \\ 1-\alpha-m, 1-\beta-m, \alpha+\beta-\frac{1}{2}\end{matrix} : \right)=\frac{(2\alpha)_m(2\beta-1)_m(\alpha+\beta-1)_m}{(\alpha)_m(\beta)_m(2\alpha+2\beta-2)_m}$$

with α and β interchanged, on the right of (2.6), we get a known result [(2), eqn. (9), p. 187].

(vii) setting $p=l=0$, $q=k=2$, $\lambda=\mu=1$, $b_1=-n$, $b_2=\rho$, $\sigma_1=-m$, $\sigma_2=\sigma$ and using Gauss's theorem, we have a known result [(2), eqn. (2), p. 185].

(viii) Taki-g $p=0$, $q=1$, $l=1$, $k=2$, $b_1=-n$, $\rho_1=a$, $\sigma_1=-m$, $\sigma_2=b$, $\mu=-1$, $\lambda=1$ and applying Gauss's theorem, we obtain a known [(6), eqn. (2), p. 125].

We may also obtain the other known results by particular choice of parameters, and using Whipple and Dixon's theorems etc.

3. Hypergeometric Transformation:

In this section we shall consider some hypergeometric transformations.

(a) With $y=x, c=d=1$, in (2,2) and (2.3), equating the coefficients of x^r, we obtain an important transformation

$$(3.1)\quad \frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_r (a_q)_r \mu^r}{(b_q)_r}\ {}_{\gamma+l+q+1}F_{p+k+\sigma}\left[\begin{matrix}\Delta(\gamma,-m),\ \rho_l,-r,\ 1-b_q-r \\ \sigma_k, \Delta(\delta,\ n+1-r\delta),\ 1-a_p-r\end{matrix};\ \frac{\lambda}{\mu}(-1)^{p-q+\delta-1}\right]$$

$$=\frac{\prod_{i=0}^{\gamma-1}\left(\frac{-m+i}{\gamma}\right)_r (\rho_l)_r \lambda^r}{(\sigma_k)_r}\ {}_{p+\delta+k+1}F_{q+\gamma+l}\left[\begin{matrix}\Delta(\delta,-n), a_p, -r,\ 1-\sigma_k-r \\ b_q,\ \Delta(\gamma,\ m+1-s\gamma)\ 1-\rho_l-r\end{matrix};\ \frac{\mu}{\lambda}(-1)^{l-k+\gamma-1}\right],$$

Special cases of (3.1) with $\delta=r=1$:

(i) Taking $l=k-1=0$, $b_1=-n$, $\sigma_1=-m$, $\mu=-z$, $\lambda=1$, we obtain a known result [(4), eqn. (3.8), p. 395].

Identities:

(ii) Substituting $p=q=l=k=2, \lambda=\mu=1, a_1=\rho_1=\alpha, a_2=\rho_2=\beta, b_1=-n, \sigma_1=-m, b_2=\frac{1}{2}+\alpha+\beta, \sigma_2=\alpha+\beta-\frac{1}{2}$, we obtain an identity

$$(3,2)\quad \left(\alpha+\beta-\frac{1}{2}\right)_r\ {}_4F_3\left(\begin{matrix}-r,\ \alpha,\ \beta,\ \frac{1}{2}-\alpha-\beta-r \\ \alpha+\beta-\frac{1}{2},\ 1-\alpha-r,\ 1-\beta-r\end{matrix};\right)$$

$$=\left(\alpha+\beta+\frac{1}{2}\right)_r\ {}_4F_3\left(\begin{matrix}-r,\ \alpha,\ \beta,\ \frac{3}{2}-\alpha-\beta-r \\ \alpha+\beta+\frac{1}{2},\ 1-\alpha-r,\ 1-\beta-r\end{matrix};\right).$$

(iii) Setting $p=q=l=k=2, \lambda=\mu=1, a_1=\alpha, a_2=\beta, b_1=-n$, $b_2=\alpha+\beta-\frac{1}{2}$, $\rho_1=\alpha$, $\rho_2=\beta-1$, $\sigma_1=-m$, $\sigma_2=\alpha+\beta-\frac{1}{2}$, we get an identity

$$(3.3)\qquad (\beta)_r\ {}_4F_3\left(\begin{matrix}-r,\ \alpha,\ \beta-1,\ \frac{3}{2}-\alpha-\beta-r \\ \alpha+\beta-\frac{1}{2},\ 1-\alpha-r, 1-\beta-r\end{matrix};\right)$$

$$=(\beta-1)_r\ {}_4F_3\left(\begin{matrix}-r,\ \alpha,\ \beta,\ \frac{3}{2}-\alpha-\beta-r \\ \alpha+\beta-\frac{1}{2},\ 1-\alpha-r,\ 2-\beta-r\end{matrix};\right).$$

(b) We start with (1.1) and expressing the hypergeometric polynomial in series, we have

$$(3.4)\qquad x^{(\delta-1)n}\ {}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),\ a_p \\ b_q\end{matrix};\mu x^c\right]$$

$$=\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_r(a_p)_r\mu^r}{r!\,(b_q)_r}x^{(\delta-1)n+cr},$$

replacing r by $n-r$, and using the formula

$$(\alpha)_{n-k}=\frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}\text{ for }0\le k\le n,$$

we obtain

$$(3.5)\qquad x^{(\delta-1)n}\ {}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),\ a_p \\ b_q\end{matrix};\mu x^c\right]=\frac{\mu^n x^{(\delta-1)n+cn}\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_n(a_p)_n}{n!\,(b_q)_n}.$$

$$\times\ {}_{q+2}F_{p+\delta}\left[\begin{matrix}-n,\ 1-b_q-n,\ 1 \\ \Delta(\delta,n+1-n\delta),\ 1-a_p-n\end{matrix};\frac{(-1)^{p-q+\delta-1}}{\mu x^c}\right],$$

where δ and n are positive integers.

Particular cases of (3.5):

(i) With $\delta=c=\mu=1$, $a_1=n+1$, $b_1=1$, $b_2=\frac{1}{2}$, we obtain a known result [(3), eqn. (6), p. 807].

(ii) Taking $\delta=c=\mu=1$, we have a known result [(4), eqn. (3.8), p. 395].

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REFERENCES

[1] Bhatt, R.C., *On the sum of the terminating hypergeometric series* ${}_4F_3(1)$, Matematiche (Catania), Vol. 20, Fasc. 2. (1965), 185—188.

[2] Erde'lyi, A., *Higher transcendental functions*, Vol. I, McGraw-Hill, New York, (1953).

[3] Fasenmyer Sister Mary Celine, *Some generalized hypergeometric polynomials*, Bull. Amer. Math. Soc. 53 (1947), 806—812.

[4] Fields, J.L. & Wimpjet, *Expansion of hypergeometric functions in hypergeometric functions*. Mathematics of Computation, Vol. 15, No. 76 (1961), 390—395.

[5] Shah Manilal, *Certain integrals involving the product of two generalized hypergeometric polynomials*, Proc. Nat. Acad., of Sciences, India, Section A, Vol. 37, part I, (1967), 79—96.

[6] Rainville, E.D., *Special functions*, MacMillan Company, New York, (1960).