NOTES ON PAIRWISE COMPACTNESS

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1. Introduction

A bitopological space (X, L_1, L_2) is a set X together with two topologies L_1 and L_2 on X. The concept of pairwise compactness was introduced independently by P. Fletcher and Y. W. Kim, and they also obtained that a pairwise compact and pairwise Hausdorff bitopological space is pairwise normal and pairwise regular. In the paper [2], R. A. Stoltenberg defined pairwise local compactness. But it is inadequate in the sense that pairwise compact does not imply pairwise local compact defined by him.

In this paper, We shall give some characterizations about pairwise compactness, pairwise countable compactness, and pairwise Hausdorffness by means of filter base and sequence. Next we shall introduce a definition of pairwise local compactness which is different from R. A. Stoltenberg's and then we obtain some characterization about it and investigate some related properties.

Unless otherwise, we shall follow the terminology and definition in J. C. Kelly [1].

2. Characterizations

DEFINITION 2.1. [2]. A cover $\mathscr{U}$ of a bitopological space (X, L_1, L_2) is called *pairwise open* if $\mathscr{U} \subset L_1 \cup L_2$ and $\mathscr{U} \cap L_1$ contains a nonvoid set, and $\mathscr{U} \cap L_2$ contains a nonvoid set.

DEFINITION 2.2. [2]. A bitopological space is called *pairwise compact* if every pairwise open cover of the space has a finite subcover.

Now we introduce pairwise closed family, pairwise filter base and pairwise net in bitopological spaces in order to find the characterization of pairwise compactness.

DEFINITION 2.3. A family $\mathscr{F}$ of nonvoid L_1-or L_2-closed sets in X is *pairwise closed* if $\mathscr{F}$ contains F_1 and F_2 such that F_1 is a L_1-closed proper subset of X and F_2 is a L_2-closed proper subset of X.

DEFINITION 2.4. A filter base $\mathscr{F}$ on X is called *pairwise filter base* if $\mathscr{F}$

contains F_1 and F_2 such that $F_1^{L_1}$ and $F_2^{L_2}$ are proper subsets of X.

DEINITION 2.5. A net $\{S_n: n\in D\}$ in X is called ***pairwise net*** if there exist m and n in D such that $\overline{\{S_k: k\geqq n\}}^{L_1}$ and $\overline{\{S_k: k\geqq m\}}^{L_2}$ are proper subsets of X.

THEOREM 2.1. *The following four statements on a bitopological space are equivalent.*

(a) (X, L_1, L_2) is pairwise compact.

(b) Each pairwise closed family of subsets of X with finite intersection property has nonvoid intersection.

(c) Each pairwise filter base (pairwise net) has at least one L_1-and L_2-accumuation point.

(d) Each pairwise maximal filter base has at least one L_1-and L_2-limit point.

PROOF. (a) implies (b): This part is obvious from the fact that the complement of pairwise closed family is pairwise open.

(b) implies (d): Let $\mathscr{F}$ be pairwise maximal filter base on X. then the family $\{F^{L_1}: F\in\mathscr{F}\}\cup\{F^{L_2}: F\in\mathscr{F}\}$ is a pairwise closed family with finite intersection property. By the hypothesis, we have $(\cap\{F^{L_1}: F\in\mathscr{F}\})\cap(\cap\{F^{L_2}: F\in\mathscr{F}\})\neq\phi$. Now let x be a member of $(\cap\{F^{L_1}: F\in\mathscr{F}\})\cap(\cap\{F^{L_2}: F\in\mathscr{F}\})$, then x is a L_1- andL_2-accumulation point of $\mathscr{F}$. The fact that $\mathscr{F}$ is a maximal filter base on X implies that x is a L_1-and L_2-limit point of $\mathscr{F}$.

(*d*) implies (c): Let $\mathscr{F}$ be any pairwise filter base on X. Then there exists a pairwise maximal filter base $\mathscr{F}'$ on X which contains $\mathscr{F}$. By the hypothesis, there is a point x such that x is a L_1-and L_2-limit point of $\mathscr{F}'$. Since $\mathscr{F}$ is contained in $\mathscr{F}'$, x is a L_1-and L_2-accumulation point of $\mathscr{F}$.

(c) implies (a): Suppose $\mathscr{U}=\{U_\alpha: \alpha\in\Gamma\}$ is a pairwise open cover of X with no finite subcover, we claim that the family $\mathscr{F}=\{X\sim(\bigcup_{\alpha\in\Lambda}U_\alpha): \Lambda=\text{a finite subset of } \Gamma,\ U_\alpha\in\mathscr{U}\}$ is a pairwise filter base on X. Since $\mathscr{U}$ is a pairwise open cover of X, there is two nonvoid member U and V of $\mathscr{U}$ such that U is a L_1-open proper subset of X and V is a L_2-open proper subset of X. Hence $X\sim U$ is a L_1-closed proper subset and $X\sim V$ is a L_2-closed proper subset of X.

From (d), we have a point y in X such that y is a L_1-and L_2-accumulation point of $\mathscr{F}$. Since $\mathscr{U}$ is a pairwise open cover of X, there exists a member U of $\mathscr{U}$ which is a L_1-open or L_2-open, say, L_1-open and contains y. Then $U\cap(X\sim U)=\phi$ which is contradict to $y=L_1$-accumulation point of $\mathscr{F}$.

In (c), the fact that the conditions of filter and net are equivalent can be found easily.

COROLLARY 2.2. *If every filter base on* (X, L_1, L_2) *has at least one* L_1*-and* L_2*-accumulation point, then* (X, L_1, L_2) *is a pairwise compact.*

Following example shows that the converse part of corollary 2.2 does not hold.

EXAMPLE 2.1. Let $X=\{x\in E^1: x\geqq 0\}$, and let L_1 be discrete topology on X and L_2 be indiscrete topology on X. Then (X, L_1, L_2) is a pairwise compact. Let $\mathscr{F}=\left\{\left(0, \frac{1}{n}\right): n=\text{positive integer}\right\}$, then $\mathscr{F}$ is a filter base and every point of X is a L_2-accumulation point of $\mathscr{F}$. On the other hand. any point of X is not a L_1-accumulation point of $\mathscr{F}$.

Following example shows that L_1-and L_2-compactness does not imply pairwise compactness.

EXAMPLE 2.2. Let X be the unit interval $[0,1]$, let L_1 be relative topology on X by usual topology and L_2 be topology with $\left\{\left[0, \frac{2}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{1}{3}, 1\right]\right\}$ as basis, then(X, L_1) and (X, L_2) are compact, but (X, L_1, L_2) is not pairwise compact (not pairwise countably compact) since $\mathscr{U}=\left[0, \frac{2}{3}\right]\cup\left\{\left(\frac{2}{3}-\frac{1}{n}, 1\right]: n\geqq 5\right\}$ is a pairwise open cover with no finite subcover.

DEFINITION 2.6. A bitopological space (X, L_1, L_2) is called *pairwise countably compact* if every countably pairwise open cover of X has a finite subcover.

We introduce pairwise sequence and pairwise infinite set in order to find characterizations of pairwise countable compactness.

DEFINITION 2.7. A sequence $\{a_n\}$ in X is called *pairwise sequence* if $\{a_n\}$ is not dense in X relative to L_1 and L_2.

DEFINITION 2.8. A subset B of X is called *pairwise infinite set* in X if B is an infinite set and B is not dense in X relative to L_1 and L_2.

THEOREM 2.3. *The following four statements on a bitopological space are equivalent.*

(a) (X, L_1, L_2) is pairwise countably compact.

(b) Each countably pairwise closed family with finite intersection property has nonvoid intersection.

(c) Each pairwise sequence in X has at least one L_1-and L_2-accumulation point.

(d) Each pairwise infinite set in X has at least one L_1-and L_2-ω-limit point.

PROOF. (a) implies (b): This part is obvious from the fact that the complement of countably pairwise closed family is countably pairwise open.

(b) implies (c): Let $\{a_n\}$ be a pairwise sequence in X. Let $B_k=\overline{\{a_n: n\geqq k\}}^{L_1}$, $B_k'=\overline{\{a_n: n\geqq k\}}^{L_2}$ for each $k\in N$, then $\{B_k: k\in N\}\cup\{B_k': k\in N\}$ is a countably pairwise closed family with finite intersection property. Hence, from (b), we have $(\cap\{B_k: k\in N\})\cap(\cap\{B_k': k\in N\}\neq\phi$. Therefore we have at least one L_1-and L_2-accumulation point of $\{a_n\}$.

(c) implies (d): Let A be any pairwise infinite set in X and let $\{a_n: n\in N\}$ be a sequence in A such that $a_n\neq a_m$ for $n\neq m$. By the hypothesis, there exists a point x in X such that x is a L_1-and L_2-accumulation point of $\{a_n\}$. Hence, for L_1-open neighborhood U of x and each $n\in N$, there exists a $a_n^k\in\{a_n\}$ which is contained in U. That is, $\{a_n^k\}\subset U$. Since $\{a_n\}$ is a sequence consisting of distinct elements, each L_1-open neighborhood of x contains infinitely many elements of A. Similarly, each L_2-open neighborhood of x contains infinitely many elements of A. Hence x is a L_1-and L_2-ω-limit point of A.

(d) implies (a): Let $\mathscr{U}=\{U_i: i\in N\}$ be a countably pairwise open cover of X with no finite subcover, then we have a sequence $\{a_n: a_n\in X\sim\bigcup_1^n U_i$ for each $n\in N$, $U_i\in\mathscr{U}\}$ such that $a_m\neq a_n$ for $m\neq n$. Then we claim that $A=\{a_n: n\in N\}$ is a pairwise infinite set. Since $\mathscr{U}$ is a pairwise open cover, we have two member U_i and U_j of $\mathscr{U}$ such that U_i is a proper L_1-open and U_j is a proper L_2-open subsets of X. Take a point $y\in U_i$, then y is not L_1-accumulation point of A since $U\cap(X\sim\bigcup_1^i U_k)=\phi$. Similarly, A is not dense in X relative to L_2-topology.

By the hypothesis, there exists a point x in X such that x is a L_1-and L_2-ω-limit point of A. We have a member U_i of $\mathscr{U}$ containing x. But $U_k\cap(X\sim\bigcup_1^i U_k)$

$=\phi$, which is contradict to $x=L_1$-and L_2-ω-limit point of A.

COROLLARY 2.4. *If every sequence has at least one L_1-and L_2-accumulation point, then (X, L_1, L_2) is a pairwise countably compact.*

The following example shows that the converse part of corollary 2.4 does not hold in general.

EXAMPLE 2.3. Let X be the nonnegative real line, and L_1 be the usual topology and $L_2=\{\phi, U\cup(x,\infty): U\in L_1, x\in X\}$, then (X, L_1, L_2) is pairwise Hausdorff and pairwise compact by [2]. Therefore (X, L_1, L_2) is a pairwise countably compact. Take a sequence $\{x_n: x_n=n$ for each $n\in N\}$, then every point of X is a L_2-accumulation point, but any point of X is not L_1-accumulation point of $\{x_n\}$.

THEOREM 2.5. *A bitopological space (X, L_1, L_2) is pairwise Hausdorff if and only if, if a filter base (net) has a L_1-limit point and a L_2-limit point, then they are the same.*

PROOF. only if: Assume there is a filter base $\mathscr{F}$ with distinct L_1-limit point x and L_2-limit point y. Then we have a L_1-neighborhood U of x and a L_2-neighborhood V of y such that $U\cap V=\phi$. Since x and y are L_1-and L_2-limit point of $\mathscr{F}$, respectively, there are two members F and F' of $\mathscr{F}$ such that $F\subset U$ and $F'\subset V$. Therefore $F\cap F'=\phi$, which is a contradiction.

if: Assume there exist distinct points x and y in X such that any L_1-neighborhood of x and any L_2-neighborhood of y have nonvoid intersection. Then $\mathscr{F}=\{U\cap V$: $U=L_1$-neighborhood of x and $V=L_2$-neighborhood of $y\}$ is a filter base which has a L_1-limit point x and a L_2-limit point y. But $x\neq y$, which is a contradiction.

COROLLARY 2.6. *Let (X, L_1, L_2) be a pairwise first countable space. Then (X, L_1, L_2) is pairwise Hausdorff if and only if, if a sequence in X has a L_1-limit point and a L_2-limit point, then they are the same.*

3. Pairwise locally compact bitopological space

Stoltenberg [4] defined that (X, L_1, L_2) is pairwise locally compact if, for each $x\in X$, there is L_i-open neighborhood U of x such that $\bar{U}^{L_i}$ is L_j-compact $(i\neq j$, $i, j=1,2)$ and showed that if (X, L_1, L_2) is paiwise Hausdorff and pairwise locally compaet in the sense of Stoltenberg, then $L_1=L_2$. To show that definition of pair-wise locally compact given by Stoltenberg is inadequate, we give an example of pairwise compact (X, L_1, L_2) which is not pairwise locally compact in the sense of R.A. Stoltenberg.

EXAMPLE 3.1. Let $X=\mathrm{N}\cup Ni$ where $i^2=-1$, N is the set of the natural numbers. Let L_1 be topology generated by $(N-F)\cup G$ where F is a finite subset of N and G is an arbitrary subset of Ni and L_2 be topology generated by $H\cup(Ni-Fi)$ where F is a finite subset of N and H is an arbitrary subset of N. Then (X, L_1, L_2) is pairwise Hausdorff and pairwise compact and (X, L_1, L_2) is not pairwise locally compact defined by Stoltenberg. Now we give another pairwise local compactness which is implied by the pairwise compactness as follows.

DEFINITION 3.1. A bitopological space (X, L_1, L_2) is called *pairwise locally compact* if, for each $x\in X$, there is L_i-open neighborhood U of x such that U^{-L_j} is pairwise compact $(i\neq j,\ i,j=1,2)$

If (X, L_1, L_2) is pairwise compact, then (X, L_1, L_2) is pairwise locally compact. But the converse is not true, and the following example is counter-example for it.

EXAMPLE 3.2. Let X be the real line and L_1 be the usual topology for X, and $L_2=\{\phi\}\cup\{U\cup(0,1): U\in L_1\}$. Then it is easy to see that (i)$L_1\neq L_2$ and (ii) (X, L_1, L_2) is not pairwise compact, and (X, L_1, L_2) is pairwise locally compact.

THEOREM 3.1. *Let (X, L_1, L_2) be pairwise Hausdorff, then (X, L_1, L_2) is pairwise locally compact if and only if, for each $x\in X$, and each L_i-open neighborhood U of x, there is a L_i-open neighborhood V of x such that $x\in V\subset\overline{V}^{L_j}\subset U$ and $\overline{V}^{L_j}$ is pairwise compact.*

PROOF. if: clear

only if: If (X, L_1, L_2) is pairwise locally compact. For each $x\in X$, there is L_i-open W such that $x\in\overline{W}\subset\overline{W}^{L_j}$, $\overline{W}^{L_j}$ is pairwise compact. By [2] Theorem 12 $\overline{W}^{L_j}$ is pairwise regular. Let U be any L_i-open neighborhood of x, then $\overline{W}^{L_j}\cap U$ is L_i-open neighborhood of x in $\overline{W}^{L_j}$. There is L_i-open neighborhood G of x in X such that $x\in G\cap\overline{W}^{L_j}\subset(G\cap\overline{W}^{L_j})^{-L_j}\cap\overline{W}^{L_j}\subset\overline{W}^{L_j}\cap U$. Let $V=G\cap W$, then V is L_i-open neighborhood of x and $V^{Lj} \subset \overline{W}^{Lj} \cap U \subset U$. Since V^{Lj} is L_j-closed in pairwise compact $\overline{W}^{Lj}$, V^{Lj} is pairwise compact from [5] Theorem 2.9.

DEFINITION 3.2. [2] (X, L_1, L_2) is pairwise completely regular iff for each L_i-closed set C and each point $x \notin C$, there exist L_i-upper semi continuous and L_j-lower semi continuous f such that $f(x)=0$ and $f(C)=1$, $0 \leqq f(x) \leqq 1$ for all $x \in X$.

THEOREM 3.2. *If (X, L_1, L_2) is pairwise locally compact and pairwise Hausdorff, then (X, L_1, L_2) is pairwise completely regular, and hence is pairwise regular.*

PROOF. Let A be L_1-closed set not containing $p \in X$, and X-A is L_1-open neighborhoodof p, then there exist L_1-open neighborhood U and V of p such that $p \in U \subset \overline{U}^{L_2} \subset V \subset \overline{V}^{L_2} \subset X\text{-}A$, and $\overline{U}^{L_2}, \overline{V}^{L_2}$ are pairwise compact By [5] Theorem 2.18., $\overline{V}^{L_2}$ is pairwise normal. By [1] Theorem, there is L_1-upper semicontinuous and L_2-lower semicontinuous g such that $g(p)=0$ and $g(V^{-L_2}-U)=1$

Let F: $X \to I$, be the map coinciding with g on $\overline{V}^{L_2}$ and identically one on $X - \overline{V}^{L_2}$. Since $\{x: F(x)<c\} = \{x: g\{x\}<c\} \in L_1$ and $\{x: F(x)>c\} = \{x: g(x)>c\} \cup (X - \overline{V}^{L_2})$, function F is L_1-upper semicontinuous and L_2-lower semicontinuous, $F(p)=0$ and $F(A)=1$. Similarly, let B be L_2-closed set not containing p, then there is L_1-lower semicontinuous and L_2-upper semicontinuous function G such that $G(p)=0$, $G(B)=1$. These facts imply that (X, L_1, L_2) is pairwise completely regular.

THEOREM 3.3 *If C is L_i-closed subset of pairwise locally compact (X, L_1, L_2), $(i=1,2)$, then C is pairwise locally compact.*

PROOF. Let x be arbitrary element of C, then there is L_i open U in X such that $\overline{U}^{L_j}$ is pairwise compact. Now we note that $U \cap C$ is L_i-open in C. Now we claim $\overline{U \cap C}^{L_j} \cap C$ is pairwise compact. Since $\overline{U \cap C}^{L_j}$ is L_j-closed subset of pairwise compact $\overline{U}^{L_j}$, $\overline{U \cap C}^{L_j}$ is pairwise compact. From the hypothesis that C is L_i-closed, $\overline{U \cap C}^{L_j} \cap C$ is L_i-closed subset of pairise compact $\overline{U \cap C}^{L_j}$. Therefore we have $\overline{U\cap C}^{L_i}\cap C$ is pairwise compact.

By the pairwise local compactness, there is L_j-open V in X such that $\overline{V}^{L_i}$ is pairwise compact. For each $x\in C$, then $V\cap C$ is L_j-open in C. Since $\overline{V\cap C}^{L_i}\cap C$ $=\overline{V\cap C}^{L_i}$ and $\overline{V\cap C}^{L_i}$ is L_i-closed in pairwise ompact $\overline{V}^{L_i}$, we have $\overline{V\cap C}^{L_i}$ is pairwise compact. Hence C is pairwise locally compact.

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