ON THE PRODUCT OF THE STRUCTURE SPACES

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From the pair (T, Z) of a semigroup T and a left ideal Z of it, K. D. Magill, Jr. constructed a topological space $\mathscr{U}(T, Z)$ called the structure space of the pair and obtained some interesting results and nice applications to certain topological spaces [1].

In this paper we are concerned with the structure space of the pair of the direct product $\Pi_{\lambda\in\Lambda}T_\lambda$ of a family of semigroups T_λ and the direct product $\Pi_{\lambda\in\Lambda}Z_\lambda$ of a family of left ideals Z_λ of T_λ, and obtain a result that if each T_λ has a left zero then the structure space of the pair $(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$ is homeomorphic to the product of the structure spaces of the pairs (T_λ, Z_λ).

DEFINITION 1. [1] Let T be a semigroup and Z a left ideal of it. A nonempty subset A of $T\times Z$ is called a *bond* if for any finite subset $\{(t_i, z_i)\}_{i=1}^{n}\subset A$, the system of equations $\{t_i x=z_i\}_{i=1}^{n}$ has a common solution x in Z. A bond which is not properly contained in any other bond is called an *ultrabond*. $\mathscr{U}(T, Z)$ denotes the set of all ultrabonds of the pair (T, Z), and will be equipped with the topology as follows:

DEFINITION 2. [1] For each $(t, z)\in T\times Z$, we let $H(t, z)=\{A\in\mathscr{U}(T, Z):(t, z)\in A\}$. The topological space which is obtained by taking $\{H(t, z):(t, z)\in T\times Z\}$ as a subbasis for the closed subsets of $\mathscr{U}(T, Z)$ is defined to be *the structure space* of the pair (T, Z).

LEMMA 1. *For each $v\in Z$, the set $\{(t, tv):t\in T\}$ is an ultrabond of (T, Z), and will be denoted by A_v.*

PROOF. Suppose there exists a bond B which properly contains A_v. Take an element (t, z) from $B-A_v$, then $tv\neq z$. But $(t, tv)\in A_v\subset B$ and the system of equations $\{tx=z, tx=tv\}$ has no common solution x in Z. This is a contradiction to the fact that B is a bond. Hence A_v is an ultrabond of (T, Z).

From lemma 1, $\{A_v:v\in Z\}$ is a subspace of $\mathscr{U}(T, Z)$. This space will be denoted by $\mathscr{R}(T, Z)$ and referred to as the realization of Z [1].

Let a family $\{(T_\lambda, Z_\lambda)\}_{\lambda\in\Lambda}$ of pairs of semigroups T_λ and its left ideals Z_λ be given. $\Pi_{\lambda\in\Lambda}T_\lambda$ and $\Pi_{\lambda\in\Lambda}Z_\lambda$ denote the direct products of T_λ's and Z_λ's respectively. We note that $\Pi_{\lambda\in\Lambda}Z_\lambda$ is also a left ideal of $\Pi_{\lambda\in\Lambda}T_\lambda$.

LEMMA 2. *Let A_λ be a bond of the pair (T_λ, Z_λ) for each $\lambda\in\Lambda$. Then the set $\{((t_\lambda)_{\lambda\in\Lambda}, (z_\lambda)_{\lambda\in\Lambda}) : (t_\lambda, z_\lambda)\in A_\lambda, \lambda\in\Lambda\}$ is a bond of $(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$, and will be denoted by $\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$.*

PROOF. Given $\{((t^i_\lambda)_{\lambda\in\Lambda}, (z^i_\lambda)_{\lambda\in\Lambda})\}^n_{i=1}\subset\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$, $\{(t^i_\lambda, z^i_\lambda)\}^n_{i=1}\subset A_\lambda$ for each $\lambda\in\Lambda$. Since A_λ is a bond, there exists an element v_λ of Z_λ such that $t^i_\lambda v_\lambda = z^i_\lambda$ for each $i=1,2,\cdots,n$. Hence $(v_\lambda)_{\lambda\in\Lambda}\in\Pi_{\lambda\in\Lambda}Z_\lambda$ and $(t^i_\lambda)_{\lambda\in\Lambda}(v_\lambda)_{\lambda\in\Lambda}=(t^i_\lambda v_\lambda)_{\lambda\in\Lambda}=(z^i_\lambda)_{\lambda\in\Lambda}$, for each $i=1,2,\cdots,n$. Therefore $\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$ is a bond.

LEMMA 3. *Let A be a bond of $(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$. Then for each $\lambda\in\Lambda$, the set $\{(t_\lambda, z_\lambda) : P_\lambda(t)=t_\lambda$ and $P_\lambda(z)=z_\lambda$ for some $(t,z)\in A\}$, which will be denoted by A_λ, where P_λ denotes the projection to λ-th coordinates, and $A\subset\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$.*

PROOF. If $\{(t^i_\lambda, z^i_\lambda)\}^n_{i=1}\subset A_\lambda$, then there exist $\{(t^i, z^i)\}^n_{i=1}\subset A$ such that $P_\lambda(t^i)=t^i_\lambda$ and $P_\lambda(z^i)=z^i_\lambda$ for each $i=1,2,\cdots,n$. Since A is a bond, there exists an element $v\in\Pi_{\lambda\in\Lambda}Z_\lambda$ such that $t^iv=z^i, i=1,2,\cdots,n$. Hence $v_\lambda=P_\lambda(v)\in Z_\lambda$ and $t_\lambda^i v_\lambda = P_\lambda(t^i)P_\lambda(v)=P_\lambda(t^iv)=P_\lambda(z^i)=z^i_\lambda$, $i=1,2,,\cdots,n$. Therefore A_λ is a bond. Clearly $A\subset\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$.

LEMMA 4. *A bond A of $(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$ is an ultrabond if and only if $A=\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$ and A_λ is an ultrabond of (T_λ, Z_λ) for each $\lambda\in\Lambda$. In this case $(\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda)_\lambda=A_\lambda$.*

PROOF. Suppose A is an ultrabond of $(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$. Then $\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$ is a bond which contains A by lemmas 2 and 3. Hence $A=\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$ by the maximality of A. If B_λ is a bond which contains A_λ, then $\textcircled{\scriptsize T}_{\mu\neq\lambda}A_\mu\textcircled{\scriptsize T}B_\lambda$ is a bond of $(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$ and contains $A=\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$. By the maximality of A, $A=\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda=\textcircled{\scriptsize T}_{\mu\neq\lambda}A_\mu\textcircled{\scriptsize T}B_\lambda$, and hence $A_\lambda=B_\lambda$. Therefore A_λ is an ultrabond of (T_λ, Z_λ). Now suppose $A=\textcircled{\scriptsize T}_{\lambda\in\Lambda}A_\lambda$ and A_λ is an ultrabond for each $\lambda\in\Lambda$ and suppose further that B is a bond of

$(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$ which contains A. Then $A=\text{Ⓣ}_{\lambda\in\Lambda}A_\lambda\subset B\subset \text{Ⓣ}B_{\lambda\in\Lambda}$ and hence $A_\lambda\subset B_\lambda$ for each $\lambda\in\Lambda$. By the maximality of $A_\lambda, A_\lambda=B_\lambda$ for each $\lambda\in\Lambda$, and hence $A=B$. Therefore A is an ultrabond of $(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$. The last statement is obvious.

Now we define a function h from $\mathscr{U}(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$ into $\times_{\lambda\in\Lambda}\mathscr{U}(T_\lambda, Z_\lambda)$ by $h(A)=(A_\lambda)_{\lambda\in\Lambda}$. Then lemma 4 asserts that h is well defined and is a bijection.

LEMMA 5. *$h(H((t_\lambda)_{\lambda\in\Lambda}, (z_\lambda)_{\lambda\in\Lambda}))=\times_{\lambda\in\Lambda}H(t_\lambda, z_\lambda)$ and $h^{-1}(\times_{\lambda\in\Lambda}H(t_\lambda, z_\lambda))=H((t_\lambda)_{\lambda\in\Lambda}, (z_\lambda)_{\lambda\in\Lambda})$, where $\times$ denotes the cartesian product.*

PROOF. The first assertion follows from the equivalent statements: $A\in H((t_\lambda)_{\lambda\in\Lambda}, (z_\lambda)_{\lambda\in\Lambda})$, $((t_\lambda)_{\lambda\in\Lambda}, (z_\lambda)_{\lambda\in\Lambda})\in A$, $(t_\lambda, z_\lambda)\in A_\lambda$ for each $\lambda\in\Lambda$. $A_\lambda\in H(t_\lambda, z_\lambda)$ for each $\lambda\in\Lambda$, and $(A_\lambda)_{\lambda\in\Lambda}\in\times_{\lambda\in\Lambda}H(t_\lambda, z_\lambda)$.

Now the second assertion follows from the equivalent statements: $h(A)=(A_\lambda)_{\lambda\in\Lambda}\in\times_{\lambda\in\Lambda}H(t_\lambda, z_\lambda)$, $A_\lambda\in H(t_\lambda, z_\lambda)$ for each $\lambda\in\Lambda$, $(t_\lambda, z_\lambda)\in A_\lambda$ for each $\lambda\in\Lambda$, $((t_\lambda)_{\lambda\in\Lambda}, (z_\lambda)_{\lambda\in\Lambda})\in\text{Ⓣ}_{\lambda\in\Lambda}A_\lambda=A$, and $A\in H((t_\lambda)_{\lambda\in\Lambda}, (z_\lambda)_{\lambda\in\Lambda})$.

THEOREM. *If each T_λ has a left zero, then the structure space of the pair $(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda)$ is homeomorphic to the product of the structure spaces of the pairs (T_λ, Z_λ), $\lambda\in\Lambda$.*

PROOF. The first assertion of lemma 5 insures the continuity of h^{-1}. If each T_λ has a left zero 0_λ, then $H(0_\lambda, 0_\lambda)=\mathscr{U}(T_\lambda, Z_\lambda)$ and hence $h^{-1}(H(t_\mu, z_\mu)\times\times_{\lambda\neq\mu}\mathscr{U}(T_\lambda, Z_\lambda))=h(H(t_\mu, z_\mu)\times\times_{\lambda\neq\mu}H(0_\lambda, 0_\lambda))=H((t_\mu, 0_\lambda)_{\lambda\neq\mu}, (z_\mu, 0_\lambda)_{\lambda\neq\mu})$ by lemma 5. From this and the fact that $\{H(t_\mu, z_\mu)\times\times_{\lambda\neq\mu}\mathscr{U}(T_\lambda, Z_\lambda):(t_\mu, z_\mu)\in T_\mu\times Z_\mu, \mu\in\Lambda\}$ forms a subbasis for the closed subsets of $\times_{\lambda\in\Lambda}\mathscr{U}(T_\lambda, Z_\lambda)$, the continuity of h follows.

COROLLARY 1. *If each T_λ has a left zero, then the realization of $\Pi_{\lambda\in\Lambda}Z_\lambda$ is homeomorphic to the product of the realizations of Z_λ's.*

PROOF. For each $v=(v_\lambda)_{\lambda\in\Lambda}\in\Pi_{\lambda\in\Lambda}Z_\lambda$, $A_v=\{(t, tv):t\in\Pi_{\lambda\in\Lambda}T_\lambda\}=\{((t_\lambda)_{\lambda\in\Lambda}, (t_\lambda v_\lambda)_{\lambda\in\Lambda}):t_\lambda\in T_\lambda, \lambda\in\Lambda\}=\text{Ⓣ}_{\lambda\in\Lambda}A_{v_\lambda}$. Hence $h(\mathscr{R}(\Pi_{\lambda\in\Lambda}T_\lambda, \Pi_{\lambda\in\Lambda}Z_\lambda))=\times_{\lambda\in\Lambda}\mathscr{R}(T_\lambda, Z_\lambda)$. Therefore they are homeomorphic by the above theorem.

Referring to corollary (2.8) of [1]: If X is a normal Hausdorff space which contains an arc, then the structure space of the pair $(S(X), Z(X))$ is the Stone-Čech compactification of X where $S(X)$ denotes the semigroup of all continuous selfmappings on X and $Z(X)$ its kernel, i.e., the set of all constant selfmappings on X: we have the following

COROLLARY 2. *If X_λ is a normal Hausdorff space which contains an arc for each $\lambda\in\Lambda$, then the product of the Stone-Čech compactifications of X_λ's is homeomorphic to the structure space of the pair* $(\Pi_{\lambda\in\Lambda}S(X_\lambda), \Pi_{\lambda\in\Lambda}Z(X_\lambda))$.

PROOF. Since $\mathscr{U}(S(X_\lambda), Z(X_\lambda))$ is the Stone-Čech compactification βX_λ of X_λ for each $\lambda\in\Lambda$, $\times_{\lambda\in\Lambda}\beta X_\lambda = \times_{\lambda\in\Lambda}\mathscr{U}(S(X_\lambda), Z(X_\lambda))$ and is homeomorphic to $\mathscr{U}(\Pi_{\lambda\in\Lambda}S(X_\lambda), \Pi_{\lambda\in\Lambda}Z(X_\lambda))$ by the above theorem.

DEFINITION 3. [2] A topological space X is called an *S^*-space* if it is T_1 and for each closed subset F of X and each point $p\in X-F$, there exists a function f in $S(X)$ and a point y in X such that $f(x)=y$ for each x in F and $f(p)\neq y$.

It was pointed out in [2] that this class of spaces includes all completely regular Hausdorff spaces which contain an arc as well as all 0-dimensional Hausdorff spaces.

Recalling the theorem (2.3) of [1] that every S^*-space X is homeomorphic to the realization $\mathscr{R}(S(X), Z(X))$ of $Z(X)$, we have the following

COROLLARY 3. *If X_λ is an S^*-space for each $\lambda\in\Lambda$, then $\times_{\lambda\in\Lambda}X_\lambda$ is homeomorphic to the realization $\mathscr{R}(\Pi_{\lambda\in\Lambda}S(X_\lambda), \Pi_{\lambda\in\Lambda}Z(X_\lambda))$ of $\Pi_{\lambda\in\Lambda}Z(X_\lambda)$.*

PROOF. $\times_{\lambda\in\Lambda}X_\lambda$ is homeomorphic to $\times_{\lambda\in\Lambda}\mathscr{R}(S(X_\lambda), Z(X_\lambda))$, which is homeomorphic to $\mathscr{R}(\Pi_{\lambda\in\Lambda}S(X_\lambda), \Pi_{\lambda\in\Lambda}Z(X_\lambda))$ by corollary 1.

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REFERENCES

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