HOMOMORPHICALLY CLOSED PARTITIONS

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1. Introduction.

The partitioning problem for classes of rings is concerned with the following: For which classes A of rings are the following two conditions satisfied?

(1) For any radical class P and for each ring $R \in A$, we have $P(R)=0$ or $P(R)=R$.

(2) Given any partition (A_1, A_2) of A, there exists a radical class which induces the partition (A_1, A_2) of A.

By partition (A_1, A_2) of a class A of rings we mean a pair of subclasses A_1 and A_2 of A such that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 \subset \{0\}$. Given any partition (A_1, A_2) of a class A of rings, we say that a radical T induces this partition if $T \cap A = A_1$ and $S(T) \cap A = A_2$ or if $T \cap A = A_2$ and $S(T) \cap A = A_1$, where $S(T)$ denotes the semi-simple class determined by T. It is clear that if A is the class of all simple rings, then condition (1) above is satisfied by A. Not immediately obvious is the fact that the class of all simple rings also satisfies condition (2) when isomorphic rings are taken to belong to the same class of the partition. This fact is proved in [1].

We wish to find conditions on a class A of rings which solve the partitioning problem when homomorphic rings belong to the same class of a given partition of A.

In what follows all rings will be taken from some universal class W of rings which has the class of all simple rings (including simple rings with zero multiplication) as a subclass. We shall employ the following notation throughout.

$R \simeq R'$ denotes the rings R and R' are isomorphic.

$I \leq R$ denotes I is an ideal of the ring R.

$I \lneqq R$ denotes $I \leq R$, but $I \neq R$.

0, depending upon the context in which it appears, denotes the ring 0, the ideal 0, or the class $\{0\}$.

$H(A)$ denotes the class of all homomorphic images of rings in A.

$L(A)$ denotes the lower radical class in W determined by the class A.

$U(A)$ denotes the upper radical class in W determined by the class A.

2. Theorems

For isomorphically closed partitions we have the following theorem.

THEOREM 1 [4]. *Let A be a class of rings satisfying the following two conditions.*

(i) *If* $R \in A$ *and* $0 = I \leq R$, *then* $I \simeq R$.

(ii) *If* $R \in A$ *and* $R/I \neq R$, *then* $R/I \not\simeq K$ *for each* $K \in A$.

Then for any radical class P, each non-zero ring in A is either P-radical or P-semi-simple. Moreover, given any partition (A_1, A_2) *of A with isomorphic rings in the same class; if* $P = U(A_1)$ *or* $P = L(A_2)$, *then each ring in* A_1 *is P-semi-simple and each ring in* A_2 *is P-radical.*

REMARK. In order that a class A of rings solve the partitioning problem, the class A must satisfy condition (1) of § 1. There are very many ordinary rings which cannot belong to a class satisfying condition (1).

EXAMPLE. Let R be a ring with a non-zero ideal I such that R/I is simple and such that I cannot be mapped homomorphically onto R/I. Let $M = \{I\}$. We claim that $R/I \notin L(M)$. By way of contradiction assume $R/I \in L(M)$. Then, since R/I is simple, we would have $R/I \in H(M)$. But this is contrary to the assumption that I cannot be mapped homomorphically onto R/I. Thus $R/I \notin L(M)$ and hence $R \notin L(M)$. However, $0 \neq I \in L(M)$ and so $0 \neq I \subset L(M)(R) \neq R$. Thus if a class A of rings is to satisfy condition (1) of § 1, then A can contain no ring of this type. In particular, A cannot contain the direct sum of two non-isomorphic simple rings. For example, $R = Z_2(+)Z_3$, where Z_i is the ring of integers modulo i for $i = 2, 3$.

Using the same form of argument as above, we can conclude that the class A cannot contain any ring R with a non-zero ideal I such that the ideal I cannot be mapped homomorphically onto any non-zero subideal of R/I.

REMARK. In view of the preceding example, for a class A to solve the partitioning problem, it is necessary that each ring R of A be contained in each of the lower radical classes determined by each of the non-zero ideals of R.

THEOREM 2. *Let A be a homomorphically closed class of rings such that* $R \in A$ *and* $0 \neq I \leq R$ *implies that I can be mapped homomorphically onto* R/I. *Then for any radical class P we have* $P(R) = 0$ *on* $P(R) = R$. *Conversely any partition* (A_1, A_2) *of A with homomorphic rings in the same class is induced by* $U(A_i)$ *for* $i = 1, 2$.

PROOF. Let P be any radical class, and let $R\in A$. If $P(R)\neq 0$, then $P(R)$ can be mapped homomorphically onto $R/P(R)$. Then $R/P(R)\in P\cap S(P)=0$, i.e., $R=P(R)$. Hence, either $P(R)=0$ or $P(R)=R$ for each $R\in A$. Next let (A_1, A_2) be any partition of A with homomorphic rings in the same class. Now $R\in A_1$ and $0\neq I\leq R$ implies I can be mapped homomorphically onto $R/I\in A_1$. Thus from Kurosh [2], we have $U(A_1)=\{R\in A: R/I\notin A_1 \text{ for all } I\lneqq R\}$. Let $R\in U(A_1)$. Then $R\notin A_1$ and so $R\in A_2$. Therefore $U(A_1)\subset A_2$. If $R\in A_2$, then $R/I\in A_2$ for each ideal I of R. Hence $R/I\notin A_1$ for each ideal I of R such that $I\neq R$. Thus $R\in U(A_i)$. It follows that $A_2=U(A_1)$ and hence that $U(A_1)$ induces the partition (A_1, A_2) of A. Similarly $U(A_2)$ induces the partitions (A_1, A_2) of A.

NOTE. A' denote the class of all simple rings, and let B denote the class of all simple rings with unity. Set $C=\{R(+)R:\ R\in B\}$. Then $A'\cup C$ is a class A satisfying the conditions of Theorem 2.

THEOREM 3. *Let A be a homomorphically closed and hereditary class of rings such that $R\in A$ and $0\neq I\leq R$ implies that I can be mapped homomorphically onto R. Then for any radical class P we have $P(R)=0$ on $P(R)=R$ for each $R\in A$. Conversely, any partition (A_1, A_2) of A with homomorphic rings in the same class is induced by $L(A_i)$ for $i=1,2$.*

PROOF. Let P be any radical and let $R\in A$. If $P(R)\neq 0$, then $P(R)$ can be mapped homomorphically onto R and so $R\in P$. Next let (A_1, A_2) be any partition of A with homomorphic rings in the same class. We shall use the lower radical construction of Lee [3]. Let $R\in L(A_1)$ and let $I\lneqq R$. Then $D(R)\cap A_1\neq 0$, i.e., we have a finite chain of subideals:

$$0\neq J_n\leq J_{n-1}\leq\cdots\cdots\leq J_1=R,\ \text{with } J_n\in A_1.$$

But A is hereditary and $R\in A$. Whence J_k can be mapped homomorphically onto J_{k-1} for $k=2,3,\cdots,n$. Since $J_n\in A_1$ and since A_1 is homomorphically closed, we have $R=J_1\in A_1$. Thus $R\in A_1$ and so $L(A_1)=A_1$. Similarly $L(A_2)=A_2$.

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