A NOTE ON SINGULARITY OF MEASURES

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In this note we shall give some characterizations of singularity and S-singularity for measures, and from these characterizations we have an interisting result about decomposition in product measures.

Following definitions are due to Johnson [1].

DEFINITIONS. Let μ and ν be measures on σ-ring $\mathcal{S}$ of subsets of X. A set A is called a *locally measurable* if $A \cap E$ is measurable for each measurable set E.

We say that μ and ν are *(mutually) singular*, and we write $\mu \perp \nu$, if there exists a locally measurable set A such that $\mu(E \cap A)=0=\nu(E-A)$ for each $E \in \mathcal{S}$. It is equivalent to the fact that there exist disjoint measurable sets B and C such that $B \cup C=E, \mu(B)=\nu(C)=0$ for each $E \in \mathcal{S}$.

We also say that ν is *S-singular* with respect to μ, denoted by $\nu S \mu$, if given $E \in \mathcal{S}$, there is a measurable $F \subset E$ such that $\mu(E)=\nu(F)$ and $\mu(F)=0$.

We shall call that μ is *absolutely continuous* with respect to ν, in symbols $\mu \langle\langle \nu$, if $\mu(E)=0$ for every measurable set E for which $\nu(E)=0$.

Other definition and terminology follow those in Halmos [7].

In order to prove our main result we have the following.

LEMMA 1. *Let μ and ν be measures on σ-ring $\mathcal{S}$ which is generated by a class $\mathcal{F}$ of sets. Then the following are equivalent:*

(a) $\mu \perp \nu$

(b) for each $E \in \mathcal{F}$, there exists a locally measurable set A such that $\mu(E \cap A)=\nu(E-A)=0$.

PROOF. (b) implies (a): Let $\mathfrak{M}=\{ E \mid$ there exist a locally measurable A such that $\mu(E \cap A)=0=\nu(E-A) \}$. Now we claim that $\mathfrak{M}$ is σ-ring. Let $\{E_n\}$ be a sequence in $\mathfrak{M}$. Then, for each n, there exist B_n and C_n such that $\mu(B_n)=\nu(C_n)=0$, $B_n \cap C_n=\phi$, and $B_n \cup C_n=E_n$.

Now let $\bigcup_n B_n=B$, $\bigcup_n C_n=C$, and $\bigcup_n E_n=E$. Then $\mu(B-C)=0$, $\nu(C)=0$, and $(B-C) \cup C=E$. Hence we have $\bigcup_n E_n=E \in \mathfrak{M}$. Clearly we know that $\mathfrak{M}$ is closed under the formation of differences. These facts imply that $\mathscr{S}\subset\mathfrak{M}$. Thus we have $\mu\perp\nu$.

(a) implies (b): This part is obvious from the meaning of generator.

Using above lemma, we have the following characterization of singularity for measures.

THEOREM 2. *Let μ_1 and μ_2 be σ-finite measures on σ-ring $\mathscr{S}$. Then the following are equivalent:*

(a) $\mu_1\perp\mu_2$.

(b) $(\mu_1\times\nu)\perp(\mu_2\times\lambda)$ for arbitrary σ-finite measures ν and λ, which are defined in the same measurable space.

PROOF. (a) implies (b): Suppose $\mu_1\perp\mu_2$, and let ν and λ be given two σ-finite measures on the same measurable space. Now let $E\times F$ be a measurable rectangle on the product space in which $\mu_1\times\nu$ and $\mu_2\times\lambda$ is defined. Then there exist B and C such that $B\cup C=E$, $B\cap C=\phi$, $\mu_1(B)=\mu_2(C)=0$. Hence we have $(B\times F)\cap(C\times F)=\phi$, $(B\times F)\cup(C\times F)=E\times F$, and $\mu_1\times\nu(B\times F)=\mu_1(B)\cdot\nu(F)=0$, $\mu_2\times\lambda(C\times F)=\mu_2(C)\cdot\lambda(F)=0$. Since the family of all measurable rectangle acts as generator in the product measure space, we have $(\mu_1\times\nu)\perp(\mu_2\times\lambda)$ from lemma 1.

(b) implies (a): Suppose$(\mu_1\times\nu)\perp(\mu_2\times\lambda)$ for any σ-finite measures ν and λ with the same domain. Now let Y be a nonvoid countable set and $\mathscr{P}(Y)$ be the family of all subsets of Y. Define a measure on $\mathscr{P}(Y)$ by $\nu(F)$=the mumber of elements of F, for each $F\in\mathscr{P}(Y)$. Then, from the hypothesis, we have $(\mu_1\times\nu)\perp(\mu_2\times\nu)$.

Let E be a member in $\mathscr{S}$ and F be a nonvoid subset of Y, then there exist B and C such that $B\cup C=E\times F, B\cap C=\phi$ and $\mu_1\times\nu(B)=0=\mu_2\times\nu(C)$, B and C in $\mathscr{S}\times\mathscr{P}(Y)$. Since $B^y=\{x\mid(x,y)\in B\}$ is a measurable set in $\mathscr{S}$ and Y is a countable set, $P[B]=\{x\mid(x,y)\in B\}=\cup\{E^y\mid y\in Y\}$ is a measurable set in $\mathscr{S}$.

In the case that $P[B]\subsetneqq E$, we have

$$0=\mu_1\times\nu(B)=\int\nu(B_x)\,d\mu_1(x)\geqq\int_{P[B]}1d\mu_1=\mu_1(P[B]),$$

and

$$0=\mu_2\times\nu(C)=\int\nu(C_x)d\mu_2(x)\geqq\int_{E-P[B]}1d\mu_2=\mu_2(E-P[B]).$$

On the other case, $P[B]=E$, we have

$$0=\mu_1\times\nu(B)=\int\nu(B_x)d\mu_1(x)\geqq\int_E 1d\mu_1=\mu_1(E).$$

These facts implies that $\mu_1\perp\mu_2$.

REMARK 3. From the fact that, if μ_1 and μ_2 are σ-finite measure on σ-ring $\mathscr{S}$, $\mu_1\perp\mu_2$ if and only if $\mu_1 S\mu_2$ [3], we can rewrite above theorem with respect to S-singular as follow:

The relation $\mu_1 S\mu_2$ is equivalent to the fact that $(\mu_1\times\nu)S(\mu_2\times\lambda)$ for any measures ν and λ with the same domain.

Thus it is obvious that singularity and S-singularity is productive. On the other hand, we can prove directly that S-singularity is productive, but it is so tedious.

Now we apply these results to a decomposition of measure in the product measure space.

THEOREM 4. *Let λ_1 and λ_2 be σ-finite measure on measurable space $(X\times Y,\ \mathscr{S}\times\mathscr{T})$ such that $\lambda_1=\mu\times\nu$, $\lambda_2=\mu'\times\nu'$ in the sence of Halmos* [7] *and $\nu'\langle\langle\nu$. Then there exist a unique decomposition $\lambda_2=(\alpha_1\times\nu')+(\alpha_2\times\nu')$ of λ_2 into the sum of $\alpha_1\times\nu'$ and $\alpha_2\times\nu'$ such that $\alpha_1\times\nu'\langle\langle\lambda_1$, $(\alpha_2\times\nu')S\lambda_1$, and $(\alpha_1\times\nu')S(\alpha_2\times\nu')$, where α_1 and α_2 are two measures on $\mathscr{S}$.*

PROOF. By Luther's result [2], there exist a unique decomposition $\mu'=\alpha_1+\alpha_2$ of μ' into the sum of α_1 and α_2 such that $\alpha_1\langle\langle\mu$, $\alpha_2 S\mu$, and $\alpha_1 S\alpha_2$. From the above remark, we have $(\alpha_2\times\nu')S(\mu\times\nu)$, $(\alpha_1\times\nu')S(\alpha_2\times\nu')$. Since $\nu'\langle\langle\nu$ and $\alpha_1\langle\langle\mu$, we have $(\alpha_1\times\nu')\langle\langle(\mu\times\nu)$ [6]. Finally we have $(\alpha_1+\alpha_2)\times\nu'=(\alpha_1\times\nu')+(\alpha_2\times\nu')$ [5]. Thus we obtain the required result.

On the contrary, if we apply above method to a decomposition of ν', the fact that a decomposition satifying the condition mentioned above exist uniquely implies the following.

COROLLARY 5. ***Under the same hypothesis as before, if we have $\nu'\langle\langle\nu$ and $\nu\langle\langle\nu'$ then we have***

(a)$(\mu'\times\beta_1)=(\alpha_1\times\nu')$ and (b)$(\mu'\times\beta_2)=(\alpha_2\times\nu')$,

where $\mu'=\alpha_1+\alpha_2$, $\alpha_1\langle\langle\mu$, $\alpha_2 S\mu$, $\alpha_1 S\alpha_2$, and $\nu'=\beta_1+\beta_2$, $\beta_1\langle\langle\nu$, $\beta_2 S\nu$, $\beta_1 S\beta_2$.

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