ROYDEN COMPACTIFICATION OF HARMONIC SPACES

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Several fundamental results in the theory of Riemann surfaces and Riemannian spaces are consequences of the presheaf property of harmonic functions and have, accordingly, been extended to the axiomatic setting of Brelot [1]. In particular, Constantinescu and Cornea [4] introduced the counterpart of the Wiener compactification and systematically developed the study of bounded harmonic functions in terms of this compactification. The purpose of the present paper is to construct a Royden-type compactification of a harmonic space and to generalize known results to the axiomatic setting.

1. Let X be a connected, locally compact, non-compact Hausdorff space, and H a family of real-valued continuous functions (harmonic functions) with open domains in X such that the class of harmonic functions on an open set forms a real linear space. The pair (X, H) is a harmonic space if the following axioms are met (Brelot [1]):

(A.1) A function u defined on an open set U is harmonic on U if and only if its restriction to any open $V \subset U$ is harmonic on V. A relatively compact open set Ω is called inner regular if for any $f \in B(\partial\Omega)$, the class of bounded functions on $\partial\Omega$, there exists a unique function $u_{f\Omega} \in B(\Omega \cup \partial\Omega)$ such that $u_{f\Omega} = f$ on $\partial\Omega$, $u_{f\Omega}$ is harmonic on Ω, and $u_{f\Omega} \geq 0$ for $f \geq 0$.

(A.2) X has a base by inner regular domains.

(A.3) The upper envelope of an increasingly directed net of harmonic functions on a domain U is either harmonic or $+\infty$ on U.

It is known (Constantinescu-Cornea [2]) that under the axioms (A.1) and (A.2), axiom (A.3) is equivalent to the following axiom:

(A.3)′ Every increasing sequence of harmonic functions on a domain U either converges to a harmonic function or diverges to $+\infty$ on U.

(A.3)″ For any domain U, a compact set $K \subset U$, and any fixed point $x_0 \in K$, there exists a constant $k = k(x_0, K, U) \geq 1$ such that $u(x) \leq ku(x_0)$ for all $x \in K$ and all nonnegative harmonic function u on U.

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In our discussion we also postulate:

(A.4) 1 belongs to the class $\overline{H}$ of superharmonic functions, that is, $1 \geq u_{1\Omega}$ on Ω for any inner regular domain $\Omega \subset X$.

(A.5) X is first countable.

By Constantinescu-Cornea [2], (A.5) implies that X has a countable base. By means of (A.2) and (A.4) it can be shown (Loeb [6]) that X has a countable exhaustion $\{\Omega_n\}$ by inner regular domains $\Omega_n \subset X$, that is, $X = \bigcup_{n=1}^{\infty} \Omega_n$ and $\Omega_n \subset \Omega_{n+1}$ for all $n \geq 1$.

2. Let μ be a Radon measure on X with $\mu(U) > 0$ for any nonempty open $U \subset X$, and $B(X)$ the space of real-valued bounded continuous functions on X. we consider a bilinear mapping $a(\cdot, \cdot)$ from $B(X)$ (not necessary defined on the entire space) into $L^1_{loc}(X, \mu)$ with the following properties (a.1)—(a.5):

(a.1) The domain $T(X)$ of $a(\cdot, \cdot)$ forms a real vector lattice under the pointwise maximum $f \cup g$ and minimum $f \cap g$, and $1 \in T(X)$.

(a.2) $a(f, f) \geq 0$ μ-*a.e.* and the equality holds on an open $U \subset X$ only if $f =$const. on U. $a(\cdot, \cdot)|U$ is bilinear on $T(X)|U$.

(a.3) $a(f, f) \geq a(f \cap \alpha, f \cap \alpha)$ μ-*a.e.* on X for all real $\alpha \geq 0$.

(a.4) For any inner regular domain $\Omega \subset X$ and $f \in T(X)$,

$$\int_\Omega a(f, f)\, d\mu = \int_\Omega a(f - u_{f\Omega}, f - u_{f\Omega})\, d\mu + \int_\Omega a(u_{f\Omega}, u_{f\Omega})\, d\mu.$$

For any $g \in T(X)$ we write

$$A(g) = \sup \left\{ \int_\Omega a(g, g)\, d\mu \,\middle|\, \Omega \text{ inner regular domain} \right\}.$$

If $A(g) < \infty$, it is the directed limit of the net $\left\{ \int_\Omega a(g, g)\, d\mu \right\}_\Omega$ as $\Omega \to X$. It is easy to see that the class $F(X) = \{f \in T(X) \mid A(f) < \infty\}$ forms a real linear space such that $A(\alpha f) = \alpha^2 A(f)$, $A(f+g)^{\frac{1}{2}} \leq A(f)^{\frac{1}{2}} + A(g)^{\frac{1}{2}}$ for all $f, g \in F(X)$ and real α. Set

$$A_\Omega(f, g) = \frac{1}{2} \int_\Omega \{a(f, g) + a(g, f)\}\, d\mu.$$

By virtue of $a(f, g) + a(g, f) = a(f+g, f+g) - a(f, f) - a(g, g)$, it is seen that the net $\{A_\Omega(f, g)\}_\Omega$ has the directed limit $A(f, g)$, say, such that

$$A(f, g) = \frac{1}{2} \{A(f+g) - A(f) - A(g)\}.$$

(a,5) Let $\{f_n\}$ be a sequence in $F(X)$ such that $\lim_n A(f_n)$ exists and $f = B - \lim_n f_n$

on X, that is, $\{f_n\}$ is uniformly bounded and converges to f uniformly on compact subsets of X. Then $f \in F(X)$ and for any $g \in F(X)$, $A(f,g) = \lim_{n\to\infty} A(f_n, g)$. Thus the space $F(X)$ is an inner product space with the inner product $A(\cdot,\cdot)$. We shall write $f = AB - \lim_n f_n$ on X if $A(f-f_n) \to 0$ and $f = B - \lim_n f_n$ on X.

If X is an orientable Riemannian manifold with a positive definite metric tensor, and μ the volume element of X, then we can define $a(f,g) = df \wedge {}^*dg$ or $df \wedge {}^*dg + Pfg$ according as the harmonic class H consists of harmonic or P-harmonic functions. Therefore the corresponding theories will be special cases of the present discussion (cf. Sario-Nakai [8], Kwon-Sario-Schiff [5]).

3. The inner product space $F(X)$ is called the Royden space associated with the harmonic space (X, H) if the subspace $F(X) \cap C_0(X)$ separates points in X.

The following propositions are immediate (cf. Constantinescu-Cornea [3], Sario-Nakai [8]).

PROPOSITION. *The Royden Space $F(X)$ is complete in the AB-topology.*

PROPOSITION. *There exists a unique (up to homeomorphisms) compactification X^* of X such that*

(i) *X^* is a compact Hausdorff space and contains X as an open dense subset,*

(ii) *every $f \in F(X)$ has a continuous extension to X^*,*

(iii) *$F(X)$ separates points in X^*.*

We shall call X^* the Royden compactification of the harmonic space (X, H) relative to A, and $\beta = X^* - X$ the Royden boundary of X.

The Stone-Weierstrass theorem yields:

PROPOSITION. *The uniform closure of the Royden space $F(X)$ is either $B(X^*)$ or $B_s(X^*)$, where $B_s(X^*) = \{f \in B(X^*) \mid f(s) = 0\}$, for some point $s \in X^*$.*

In the latter case such a point s is unique and belongs to the Royden boundary β since the class $F(X) \cap C_0(X)$ separates points in X. If s exists, it is called the singular point of X^*.

The singular point s exists if and only if $1 \notin F(X)$.

4. Denote by $F_\delta(X)$ the AB-closure of the space $F(X) \cap C_0(X)$. Cleary $F_\delta(X)$ is a linear subspace of $F(X)$; we will call it the Royden potential subspace.

LEMMA. *If* $f \in F_{\delta}(X)$ *and* $u \in HF = H \cap F(X)$, *then* $A(f, u) = 0$.

PROOF. Choose a sequence $\{f_n\}$ in $F(X) \cap C_0(X)$ such that $f = AB\text{-}\lim_n f_n$ on X. Since $|A(f, u) - A(f_n, u)|^2 \leqq A(f - f_n) \cdot A(u)$, $A(f, u) = \lim_n A(f_n, u)$. For any inner regular Ω with $\Omega \supset \operatorname{supp} f_n$, $A(f_n, u) = \int_{\Omega} a(f_n, u) d\mu = 0$ by (a, 4), and the assertion follows.

The compact set $\delta = \{x \in X^* | f(x) = 0$ for all nonnegative $f \in F_{\delta}(X)\}$ plays an important role in the study of harmonic functions with finite A-norms. Clearly $\delta \subset \beta$, and $s \in \delta$ if the singular point s exists. The set δ is called the Royden harmonic boundary of X.

THEOREM. *If the harmonic boundary* δ *of* X *is empty, then the class* $HF(X)$ *consists of constants only.*

PROOF. Since $\delta = \phi$, there exists a function $f \in F_{\delta}(X)$ such that $f \geq 1$ on X^*. Choose a sequence $\{f_n\}$ in $F(X) \cap C_0(X)$ with $f = AB - \lim_n f_n$ on X.

Let K be an outer regular compact subset of X and $\{\Omega_n\}$ an exhaustion of X by inner regular domains Ω_n with $K \subset \Omega_n$ for all $n \geq 1$ (cf. Loeb [6]). Construct $w_n \in F(X)$ such that w_n is harmonic on Ω_n and $w_n = f$ on $X - \Omega_n$. Since $\infty > A(f) = A(w_n) + A(f - w_n)$ and $A(w_{n+p}) = A(w_n) + A(w_{n+p} - w_n) \geq A(w_n)$ the sequence $\{w_n\}$ is A-Cauchy on X. Furthermore it is uniformly bounded on X, and in view of $(A.3)''$ we may assume that $w = B - \lim_n w_n$ exists on R. Thus $w = AB - \lim_n w_n$ on X and $w \in HF(X)$. Observe that f and f-w are in the class $F_{\delta}(X)$. Thus $w \in F_{\delta}(X) \cap HF(X)$ and w=constant by the above lemma.

If $w = 0$, $HF(X) = \{0\}$ as desired. In the case $w \neq 0$ note that $A(1) = 0$ and $1 \in F_{\delta}(X) \cap HF(X)$.

Let ω_n be such that ω_n is harmonic on $\Omega_n - K$, $\omega_n = 0$ on K, and $\omega_n = 1$ on $X - \Omega_n$. As above we conclude that $\omega = AB - \lim_n \omega_n$ exists on X and ω is harmonic on $X - K$. Since $A(\omega) = A_{X-K}(\omega) = \lim_{n \to \infty} A_{X-K}(\omega_n, \omega) = 0$, $\omega \equiv 0$ on X. For any $u \in HF(X)$ and $n \geq 1$, we have $|u| \leq \max_K |u| + (\sup_X |u|)\, \omega_n$ on Ω_n and therefore $|u| \leq \max_K |u|$ on X. Thus one of the nonnegative harmonic functions $\max_K |u| \pm u$ takes its mininum in X and therefore must be constant.

This completes the proof of the theorem.

REMARK. The converse of the theorem is not true in general.

5. In the above proof we have shown that if $\delta=\phi$, then either $HF(X)=\{0\}$, or $HF(X)=\{\text{const.}\}$ and $1\in F_{\delta}(X)$. In the present paper we understand that $HF(X)=\{0\}$ whenever $\delta=\phi$.

THEOREM. *The following direct sum decomposition is valid*:

$$F(X)=HF(X)\oplus F_{\delta}(X).$$

PROOF. Clearly it suffices to consider nonnegative functions $f\in F(X)$.

For an exhaustion $\{\Omega_n\}$ of X by inner regular domains, construct $u_n\in F(X)$ such that u_n is harmonic on Ω_n and $u_n=f$ on $X-\Omega_n$. Since $\{u_n\}$ is uniformly bounded on X we may assume that $u=B-\lim_n u_n$ exists on X and u is harmonic on X. By virtue of $A(u_n)=A(u_{n+p})+A(u_n-u_{n+p})$, we have $u=AB-\lim_n u_n$ on X and $u\in HF(X)$. Thus $f=u+(f-u)$ is a desired decomposition. Since $F_{\delta}(X)\cap HF(X)=\{0\}$ the uniqueness of the decomposition is trivial.

COROLLARY. *If* $f\geq 0$, then $u\geq 0$ on X.

COROLLARY. $F(X)=F_{\delta}(X)$ if $\delta=\phi$. *Conversely, if* $F(X)=F_{\delta}(X)$, *then either* $\delta=\phi$ *or* $\{s\}$ *according as the singular point does not or does exist.*

6. We now turn to the behavior of HF-functions near the harmonic boundary δ. The importance of the harmonic boundary lies in that every nonconstant HF-function takes its absolute maximum on it.

THEOREM. *If a function* $u\in HF(X)$ *has the property* $|u|\leq M$ *on the harmonic boundary* δ, *then the same inequality is valid on the entire space* X. *Thus* $|u|\leq \max_{\delta}|u|$ *on* X *for all* $u\in HF(X)$.

PROOF. It suffices to show that $u\leq M$ on X whenever $u\leq M$ on δ. By our convention $HF(X)=\{0\}$ for $\delta=\phi$ we can assume that $\delta\neq\phi$.

For any $\varepsilon>0$, set $E=\{x\in X^*|u(x)\geq M+\varepsilon\}$. Since E is compact and $E\cap\delta=\phi$, there exists a function $h\in F_{\delta}(X)$ such that $h\geq 0$ on X^* and $h\geq 1$ on E. For an exhaustion $\{\Omega_n\}$ of X by inner regular domains Ω_n, construct $v_n\in F(X)$ such that v_n is harmonic on Ω_n and $v_n=h$ on X-Ω_n. It is not difficult to see that $v=AB\text{-}\lim_n v_n$ exists on X and $v\in F_{\delta}(X)\cap HF(X)$. Therefore $v\equiv 0$ on X. Since $1\in$

$\overline{H}$ and $M+\varepsilon-u+\|u\|_\infty h \geq 0$ on X, we have $M+\varepsilon-u+\|u\|_\infty v_n \geq 0$ on Ω_n for all $n\geq 1$ and therefore $u\leq M+\varepsilon+\|u\|_\infty v_n$ on Ω_n. On letting $n\to\infty$ and then $\varepsilon\to 0$ we obtain the desired inequality.

COROLLARY. *Every function in $HF(X)$ vanishes identically on X whenever it does so on the harmonic boundary.*

7. We have seen that every $f\in F(X)$ has the unique decomposition $f=u+g$ for some $u\in HF(X)$ and $g\in F_\delta(X)$, and $u\geq 0$ whenever $f\geq 0$. Since the class $F(X)$ is uniformly dense in $B_s(X^*)$, it is natural to ask whether the HF-projection $f\to u$ can be extended to $B_s(X^*)$.

We claim:

THEORFM. *There exists a positive bounded linear operator $\pi: B_s(\delta)\to HB(X)$ such that $|\pi(f)|\leq\max_\delta|f|$ on X. Here $HB(X)=H\cap B(X)$.*

PROOF. By Tietze's extension theorem every $f\in B_s(\delta)$ has a continuous extension $\bar{f}$ to X^* with $\max_{X^*}|\bar{f}|=\max_\delta|f|$. Choose $f_n\in F(X)$ such that $\max_{X^*}|\bar{f}-f_n| <1/n$, and let u_n be the HF-projection of f_n on X. Then $\max_{X^*}|u_n-u_m|=\max_\delta|u_n -u_m|<\frac{1}{n}+\frac{1}{m}$. Therefore there exists a harmonic function $u\in B(X^*)$ with $\max_{X^*}|u-u_n|\leq 1/n$, and such a u is uniquely determined by f.

Set $\pi(f)=u$. Clearly π is well-defined and has the property $|\pi(f)|\leq\max_\delta|f|$ on X. Thus π is a bounded linear operator from $B_s(\delta)$ into $HB(X)$. Assume that $f\in B_s(\delta)$ is nonnegative. Given any $\varepsilon>0$ choose N such that $|\pi(f)-u_n|<\varepsilon$ on X^* for all $n\geq N$. In particular on δ $u_n\geqq\pi(f)-\varepsilon=f-\varepsilon\geqq-\varepsilon$ and therefore $u_n\geqq-\varepsilon$ on X by the proof of Theorem 6. Letting $n\to\infty$ and then $\varepsilon\to 0$ we have the positiveness of the operator π.

For a fixed point $x_0\in X$, consider the functional $L_{x_0}(f)=(\pi f)(x_0)$ on $B_s(\delta)$.

Clearly L_{x_0} belongs to the topological dual $B_s(\delta)'$ of $B_s(\delta)$. By the Hahn-Banach theorem, L_{x_0} has a continuous extension $\bar{L}_{x_0}\in B(\delta)'$. The Riesz representation theorem yields a regular (signed) Borel measure $\mu=\mu_{x_0}$ on δ such that

$$L_{x_0}(f)=\int_{\bar{\partial}} f d\mu \quad \text{for all } f\in B(\bar{\partial}).$$

In particular

$$(\pi f)(x_0)=L_{x_0}(f)=\int_{\bar{\partial}} f d\mu \text{ for every } f\in B_s(\bar{\partial}).$$

On adjusting μ by $\mu(s)=$o if the singular point s exists, we obtain:

THEOREM. *For a given $x_0\in X$ there exists a unique nonnegative Borel measure* $\mu=\mu_{x_0}$ on $\bar{\partial}$ with $\mu(s)=0$ such that

$$(\pi f)(x_0)=\int_{\bar{\partial}} f d\mu \quad \textit{for every } f\in B_s(\bar{\partial}).$$

By virture of (A. 3)″ we have:

THEOREM. *There exists a nonnegative function $P(x,t)$ on $X\times\bar{\partial}$ such that*
(i) *$P(x_0,t)=1$ on $\bar{\partial}$, and for each $x\in X$, $P(x,t)$ is Borel measurable on $\bar{\partial}$,*
(ii) *$P(x,t)$ is essentially bounded on $\bar{\partial}$, uniformly on every compact subset of X,*
(iii) *$(\pi f)(x)=\int_{\bar{\partial}} P(x,t)f(t)d\mu(t)$ on X for every $f\in B_s(\bar{\partial})$.*

We call $\mu=\mu_{x_0}$ the HF-measure on $\bar{\partial}$ centered at $x_0\in X$, and $P(x,t)$ the HF-kernel on $X\times\bar{\partial}$.

COROLLARY. *A function u belongs to the class $HF(X)$ if and only if $u(x)=\int_{\bar{\partial}} P(x,t)\ f(t)\ d\mu(t)$ on X for some $f\in F(X)$. In this case $u=f$ on $\bar{\partial}$.*

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REFERENCES

[1] M. Brelot, *Lectures on potential theory*, Tata Inst. Fund, Res., Bombay, 1960, 170 pp.

[2] C. Constantinescu-A. Cornea, *On the axiomatic functions*, Ann. Inst. Fourier 13(1963), 373—388.

[3] ____________________, *Ideale Ränder Riemannscher Flächen*, Springer, 1963, 244 pp.

[4] ____________________, *Compactifications of harmonic spaces*, Nagoya Math. J. 25 (1965), 1—57.

[5] Y. K. Kwon - L. Sario - J. Schiff, *Bounded energy-finite solutions of $\Delta u = Pu$ on a Riemannian manifold*, Nagoya Math. J. Vol. 42, (1971) 95—108.

[6] P. Loeb, *An axiomatic treatment of pairs of elliptic partial differential equations*, Ann. Inst. Fourier 16(1966) 167—288.

[7] M. Nakai, *A measure on the harmonic boundary of a Riemann surface*, Nagoya Mah. J. 17(1960) 181—218.

[8] L. Sario - M. Nakai, *Classification of Riemann surfaces*, Springer, 1970, 446 pp.