On the semi-developable spaces

CHINKU CHU

The notation and terminology used in this paper are to follow those of J. Dugundji [2] mainly and any mapping is to be continuous surjective. We shall assume that the terminology of neighborhood is used in the general sense, not necessarily open. We define semi-developable spaces due to Charles C. Alexander [1] as follows.

Let $\Delta=\{g_n|n\in N\}$ be a sequence of (not necessarily open) covers of a space X, where N is the set of positive integers. Δ is a *semi-development* for X if and only if for each $x\in X$, $\{\mathrm{St}(x,g_n)|n\in N\}$ is a neighborhood base at x. A space X is called *semi-developable* if and only if there exists a semi-development for X.

A mapping $f: X\rightarrow Y$ is *pseudo-open* if for each $y\in Y$ and any open neighborhood U of $f^{-1}(y)$, it follows that $y\in Int[f(U)]$.

For later use, we note that semi-developability is hereditary and every semi-developable space has a semi-development $\{g_n|n\in N\}$ having the property $g_{n+1}<g_n$ (g_{n+1} is a refinement of g_n) for each $n\in N$. Semi-developments having this property shall be called *refining semi-developments*. We may assume that every semi-developable space has a refining semi-development [1].

We shall use the following results due to Charles C. Alexander and M. Henry for some parts of our discussion:

(A) A space X is semi-metrizable if and only if it is a semi-developable T_0-space [1].

(B) The image of a semi-metric space under an open finite to one mapping is semi-metrizable [3].

Now we prove the following.

THEOREM 1. *If a space X is the union of a locally finite family of open semi-developable subspaces, then X is also semi-developable.*

Proof. Let $X=\bigcup\{A_i|i\in I\}$ be the union of a locally finite family of open semi-developable subspaces and $\Delta_i=\{g^i_n|n\in N\}$ be a refining semi-development for the space A_i for each $i\in I$. Consider the set $\Delta=\{g_n|n\in N\}$, where g_n is the union $\bigcup\{g^i_n|i\in I\}$. We shall show that Δ is a semi-development for X. Each $g_n\in\Delta$ is a cover of X since every g^i_n, $i\in I$, is a cover of A_i. Suppose that U_x is any neighborhood of an arbitrary point $x\in X$. Since each A_i, $i\in I$, is open in X and $\{A_i|i\in I\}$ is a locally finite family, there is a neighboorhod v^i_x of x such that $v^i_x=\mathrm{St}(x,g^i_{n_i})$ for some $n_i\in$ N and $v^i_x\subset U_x$ for $x\in A_i, i\in I$. Hence $\mathrm{St}(x,g_n)\subset U_x$ where $n=\mathrm{Max}\{n_i|x\in A_i\}$. Therefore $\Delta=\{g_n|n\in N\}$ is a semi-development for the space X.

COROLLARY. *If X is the union of a locally finite family of open semi-metric subspaces, then X is a semi-metric space.*

Proof. It clear that X is a T_0-space. But it is semi-developable by (A) and hence semi-metric.

THEOREM 2. *The product space $X=\Pi\{X_\alpha|\alpha\in\mathscr{A}\}$ of topological spaces is semi-*

Received by the editors Oct. 30, 1971

developable if and only if $\aleph(\mathscr{A})\leq\aleph_0$ and each space $X_\alpha, \alpha\in\mathscr{A}$, *is semi-developable.*

Proof. Suppose that $\aleph(\mathscr{A})>\aleph_0$. Then X is not first countable. But the semi-developable space X is first countable. Hence $\aleph(\mathscr{A})\leq\aleph_0$. The semi-developability is hereditary [1]. Thus each X_α, $\alpha\in\mathscr{A}$, is semi-developable.

Now conversely we consider firstly the case $\aleph(\mathscr{A})=\aleph_0$, where we can set $\mathscr{A}=N$. Let X be a semi-developable space with $\Delta_i=\{g^i_n|n\in N\}$ as semi-development for each $i\in N$.

Set $g_{im,\dots,jn}=\{\langle G_{im}\rangle\cap\dots\cap\langle G_{jn}\rangle|G_{im}\in g^i_m,\dots,G_{jn}\in g^j_n\}$ for a finite set $\{i,\dots,j\}\subset N$ of distinct elements and $\{m,\dots,n\}\subset N$, where $\langle G_{im}\rangle=P_i^{-1}(G_{im})$ is a slab in X.

We will construct a semi-development for X as following. Let $\Delta=\{g_{im,\dots,jn}|i,\dots,j, m,\dots,n\in N\}$ where $\{i,\dots,j\}$ is a finite set of distinct elements of N. The set Δ is a countable set of covers of X. Consider a fixed $g_{im,\dots,jn}$. For any $x\in X$, there exists $\langle G_{pq}\rangle$ such that $x\in\langle G_{pq}\rangle$ and $G_{pq}\in g^p_q$ for each $p=i,\dots,j, q=m,\dots,n$. Thus $x\in\langle G_{im}\rangle\cap\dots\cap\langle G_{jn}\rangle\in g_{im,\dots,jn}$. Hence $\bigcup g_{im,\dots,jn}=\bigcup\{\langle G_{im}\rangle\cap\dots\cap\langle G_{jn}\rangle|G_{im}\in g^i_m,\dots,G_{jn}\in g^j_n\}=X$.

For any neighborhood U_x of $x\in X$, there exists a neighborhood V_x of x such that $V_x=\langle G_i\rangle\cap\cdots\cap\langle G_j\rangle\subset U_x$, where $G_i\subset X_i,\dots,G_j\subset X_j$ are neighborhoods of $x_i,\dots,x_j$, respectively with $P_i(x)=x_i,\dots,P_j(x)=x_j$. Since each $X_k, k\in N$, is semi-developable, there exist $\mathrm{St}(x_i,g^i_m)\subset G_i,\dots,\mathrm{St}(x_j,g^j_m)\subset G_j$.

Hence $\langle\mathrm{St}(x_i,g^i_m)\rangle\cap\dots\cap\langle\mathrm{St}(x_j,g^j_m)\rangle\subset V_x\subset U_x$. Clearly $\langle\mathrm{St}(x,g^i_m)\rangle\supset\mathrm{St}(x,g_{im,\dots,jn}),\dots\langle\mathrm{St}(x_j,g^j_n)\rangle\supset\mathrm{St}(x,g_{im,\dots,jn})$.

Hence $\mathrm{St}(x,g_{im,\dots,jn})\subset\langle\mathrm{St}(x_i,g^i_m)\rangle\cap\dots\cap\langle\mathrm{St}(x,g^j_n)\rangle$ and therefore $\mathrm{St}(x,g_{im,\dots,jn})\subset U_x$. We shall show that $\mathrm{St}(x,g_{im,\dots,jn})$ is a neighborhood of x. For any point $a\in\langle\mathrm{St}(x_i,g^i_m)\rangle\cap\dots\cap\langle\mathrm{St}(x_j,g^j_n)\rangle$, there exist $G_i,\dots,G_j$ with $a_i,x_i\in G_i\in g^i_m,\dots, a_j,x_j\in G_j\in g^j_n$. Hence $a,x\in\langle G_i\rangle\cap\dots\cap\langle G_j\rangle\subset\mathrm{St}(x,g_{im,\dots,jn})$. Hence $\langle\mathrm{St}(x_i,g^i_m)\rangle\cap\dots\cap\langle\mathrm{St}(x_i,g^j_n)\rangle\subset\mathrm{St}(x,g_{im,\dots,jn})$ and then $\mathrm{St}(x,g_{im,\dots,jn})$ is a neighborhood of x since $\langle\mathrm{St}(x_i,g^i_m)\rangle\cap\dots\cap\langle\mathrm{St}(x_j,g^j_n)\rangle$ is a neighborhood of x. Hence Δ is a semi-development for X. For the second case $\aleph(\mathscr{A})<\aleph_0$, is clear the sufficiency

THEOREM 3. *If X is semi-developable and $f: X\to Y$ is a pseudo open finite to one mapping, then Y is semi-developable.*

Proof. Let $\Delta=\{g_n|n\in N\}$ be a semi-development for X. Set $\Delta'=\{f(g_n)|n\in N\}$, where $f(g_n)=\{f(G)|G\in g_n\}$. For any neighborhood U_y of a point $y\in Y$, $f^{-1}(U_y)$ is a neighborhood of $f^{-1}(y)$. Since f is finite to one mapping, we can write $f^{-1}(y)=\{x_1,x_2,\dots,x_k\}$. There exists $n\in N$ such that $\mathrm{St}(x_i,g_n)\subset f^{-1}(U_y)$, for all $i=1,2,\dots,k$. If $a\in\mathrm{St}(y,f(g_n))$, there exists $f(G)$ such that $a,y\in f(G)$ and $G\in g_n$. But $G\subset\mathrm{St}(f^{-1}(y),g_n)$, and hence $a\in f(G)\subset f[\mathrm{St}(f^{-1}(y),g_n)]$. Conversely, if $b\in f[\mathrm{St}(f^{-1}(y),g_n)]$ there is $G\in g_n$ such that $b,y\in f(G)$ and $G\in g_n$. Hence $b\in\mathrm{St}(y,f(g_n))$. Therefore we obtained $\mathrm{St}(y,f(g_n))=f[\mathrm{St}(f^{-1}(y),g_n)]$. Since $\mathrm{St}(f^{-1}(y),g_n)\subset f^{-1}(U_y), f[\mathrm{St}(f^{-1}(y),g_n)]=\mathrm{St}(y,f(g_n))\subset U_y$. The fact that f is psedo-open and $\mathrm{St}(f^{-1}(y),g_n)$ is a neighborhood of $f^{-1}(y)$ implies that $\mathrm{St}(y,f(g_n))$ is a neighborhood of y. Thus Δ' is a semi-development for Y.

COROLLARY. *The image of a semi-metric space under a psudo-open finite to one mapping is semi-metrizable.*

Proof. Let X be a semi-metric space and $f: X \to Y$ be a pseudo-open finite to one mapping. Since X is T_0 and semi-developable, Y is T_0 and semi-developable. Hence Y is a semi-metric space by (A).

REMARK. The above corollary is a generalization of the (B) which has been proved by M. Henry through some different methods used in the semi-stratifiable spaces.

References

[1] Charles C. Alexander, *Semi-developable spaces and quotient images of metric spaces,* Pacific J. of Math., Vol.37, No. 2 (1971), 277-293.

[2] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston(1967).

[3] Michael Henry, *Stratifiable spaces, Semi-stratifiable spaces, and their relaticn through mappings.* Pacific J. of Math., Vol. 37, No, 3(1971), 697-700.

Choongnam University