Methods of introducing the concept of proximity

SANG SHIN LEE

0. Introduction

The theory of Proximity spaces was essentially discovered in the early 1950's by Efremovic when he axiomatically characterized the proximity relation "A is near B (Notation: $A\delta B$)" for subsets A and B of any set X. The set X with this binary relation on the power set of X is called a proximity space, which is naturally a generalization of a metric space. For looking back to the defining axioms, one is sure that those are theorems in a pseudo-metric space if the relation "A is near B" is replaced by "The distance from A to B is zero."

Considering this proximity space with other spaces concerning nearness relation, like topological spaces and uniform spaces, one will come to the following conclusions.

First the proximity space can be induced from a uniform space $(X, \mathcal{U})$ if one defines "$A\delta B$ if and only if $U[A]\cap U[B]\neq\phi$ for every U in $\mathcal{U}$."

Secondly, with a $\bar{A}$ of every subset A of a set X as the subset $\{x : x\delta A\}$ the mapping — on the power set of X is a Kuratowski closure operator on X. Thus a proximity relation induces a topology on the set and moreover this topology is completely regular.

In this paper two new methods of introducing a proximity relation based on the idea of the preceding argument are obtained.

1. Proximity spaces

DEFINITION. A binary relation δ on the power set of X is called an (Efremovič) proximity on X if and only if δ satisfies the axioms P1—P5. The pair (X, δ) is called a proximity space.

P 1. $A \delta B$ implies $B \delta A$.

P 2. $(A\cup B) \delta C$ if and only if $A \delta C$ or $B \delta C$.

P 3. $A \delta B$ implies $A \neq \phi$, and $B \neq \phi$.

P 4. $A \bar{\delta} B$ implies there exists a subset E of X such that $A \bar{\delta} E$ and $(X\text{-}E) \bar{\delta} B$.

P 5. $A\cap B \neq \phi$ implies $A \delta B$.

It can be shown that a proximity on X induces a topology $\mathcal{T}=\mathcal{T}(\delta)$ on X if one defines the closure $\bar{A}$ of A to be the set $\{x: x\delta A\}$ and furthermore this topology is always completely regular. Conversely from a completely regular topology one can induce a compatible proximity.

We give two definitions, one for the compare of two proximities and the other for proximal neighbourhoods.

If δ_1 and δ_2 are two proximities on a set X we define δ_1 is finer than δ_2 (Notation: $\delta_1>\delta_2$) if and only if $A\delta_1 B$ implies $A\delta_2 B$. The one of the easy consequences is that the finer proximity induces the finer topology and the finer completely regular topology

Received by the editors Oct. 30, 1971

induces the finer associate proximity.

A subset B of a proxmity space (X, δ) is a proximal neighbourhood (δ-neighbourhood) of a subset A of X (in symbol $A \ll B$) if and only if $A \bar{\delta} (X-B)$. Then the relation satisfies the following properties:

N 1. $X \ll X$.

N 2. $A \ll B$ implies $A \subset B$.

N 3. $A \subset B \ll C \subset D$ implies $A \ll D$.

N 4. $A \ll B_i$ for $i=1, \ldots\ldots, n$ if and only if $A \ll \bigcap_{i=1}^{n} B_i$.

N 5. $A \ll B$ implies $(X-B) \ll (X-A)$.

N 6. $A \ll B$ implies there is C such that $A \ll C \ll B$.

On the other hand if a binary relation $\ll$ on the power set of a set X satisfies the conditions N1 - N6 then it induces a proximity relation on X defined by

$A \bar{\delta} B$ if and only if $A \ll (X - B)$.

2. A set endowed by a family of coverings

DEFINITION: A family of coverings $\Phi = \{\mathcal{U}_\alpha : \alpha \in \mathcal{A}\}$ of a set X is uniformizing, if it satisfies the following two conditions;

1) For every two members of Φ, there exists a common refinement of them in Φ.

2) Each member of Φ has its barycentric refinement in Φ.

Instead of defining a uniform structure on X in terms of subsets of $X \times X$, Tukey provided an alternate description of a uniform structure using covers of X based on the above definition. In other words a uniform structure can be induced by a uniformizing family of coverings, and vice versa, so the theorem.

THEOREM(2-1) (1) *If we let* $V_\alpha = \bigcup \{U \times U : U \in \mathcal{U}_\alpha\}$ *for all* $\mathcal{U}_\alpha \in \Phi$, *then* $u(\Phi) = \{V_\alpha : \alpha \in \mathcal{A}\}$ *is a base of a uniformity on* X.

(2) *For a uniformity* $\mathcal{U}$ *on* X, *let* $\mathcal{U}(V) = \{V[x] : x \in X\}$ *be the covering. Then the family* $c(\mathcal{U}) = \{\mathcal{U}(V) : V \in \mathcal{U}\}$ *is a uniformizing family and consequently the uniformity* $u(c(\mathcal{U}))$ *is equivalent to* $\mathcal{U}$.

REMARK: *The family* $\{St(x, \mathcal{U}_\alpha) : x \in X$ *and* $\mathcal{U}_\alpha \in \Phi\}$ *consists a base of the topology on* X *induced by the uniformity* $u(\Phi)$.

We have seen that a uniformity can be introduced by means of a uniformizing family of coverings. But since for every uniformity $\mathcal{U}$ there is an associated proximity $\delta = \delta(\mathcal{U})$ defined by

$$A \delta B \text{ if and only if } (A \times B) \cap U \neq \phi \text{ for every } U \in \mathcal{U}.$$

It seems quite natural to determine a proximity by means of a suitable family of coverings. Theorem (2-1) shows a possibility in that sense and the following theorem.

THEOREM(2-2) *Let* $\Phi = \{\mathcal{U}_\alpha : \alpha \in \mathcal{A}\}$ *be a family of coverings of a set* X *such that for every* $\alpha \in \mathcal{A}$, $\mathcal{U}_\alpha$ *has a barycentric refinement in* Φ. *Let* δ *be a relation on the power set of* X *defined by* $A \delta B$ *if and only if there exists a* U *in* $\mathcal{U}_\alpha$ *such that* $(A \times B) \cap (U \times U) \neq \phi$ *for every* $\alpha \in \mathcal{A}$. *Then* δ *is a proximity relation on* X.

NOTATION: Denote $\delta(\Phi)=\delta$ which is defined in the above theorem.

Proof: The conditions P1, P2, P3, and P5 are straightforward.

Suppose $A\bar{\delta}B$. Then there exists an $\alpha\in\mathcal{A}$, such that for each U in $\mathcal{U}_\alpha$, $(A\times B)\cap(U\times U)=\phi$. Since $\mathcal{U}_\alpha$ has a barycentric refinement, say $\mathcal{U}_\beta$, for every y in X, $St(y,\mathcal{U}_\beta)\subset U(x)$ for some U in $\mathcal{U}_\alpha$. Let $E=\bigcup_{y\in B}St(y,\mathcal{U}_\beta)$. Then $A\bar{\delta}E$. For suppose $(A\times E)\cap(V\times V)\neq\phi$ for each $V\in\mathcal{U}_\beta$. Then for every (a,b) in $(A\times E)\cap(V\times V)$ for any V in $\mathcal{U}_\beta$, there is a y in B such that b belongs to $St(y,\mathcal{U}_\beta)$. Then a, b, and y belong to $St(b,U_\beta)$ which is contained in some U in $\mathcal{U}_\alpha$. This leads a contradiction to $A\bar{\delta}B$, for some U in $\mathcal{U}_\alpha$, (a,y) is in $(A\times B)\cap(U\times U)$. If (a,b) is in$((X-E)\times B)\cap(V\times V)$ for any V in $\mathcal{U}_\beta$, then a is not in $St(b,\mathcal{U}_\beta)$ and then for this V in $\mathcal{U}_\beta$, (a,b) is not in $V\times V$, which leads a contradiction. Thus the theorem is proved.

THEOREM (2-3) *Using the notations of the preceding theorems, $\delta(\Phi)$ is equivalent to $\delta(u(\Phi))$.*

REMARK: *Note that the condition for the family Φ to have a common refinement for every pair of members of Φ in itself is not necessary in theorem* (2-2).

THEOREM(2-4) *Let $(X,\delta(\Phi))$ be a proximity space, where $\Phi=\{\mathcal{U}_\alpha : \alpha\in\mathcal{A}\}$ is a family of coverings of X satisfying that every member of Φ has a barycentric refinement in Φ. Then $A\ll B$ if and only if there is a $\mathcal{U}_\alpha$ in Φ such that $St(A,\mathcal{U}_\alpha)\subset B$.*

Proof: $A\ll B$ if and only if $A\bar{\delta}(X\text{-}B)$ if and only if there is an α in $\mathcal{A}$ such that

$$(A\times(X\text{-}B))\cap\bigcup_{U\in\mathcal{U}_\alpha}(U\times U)=\phi.$$

(Necessity) For every x in $St(A,U_\alpha)=\bigcup\{St(y,\mathcal{U}_\alpha)\ ;\ y\in A\}$ there is a y in A such that $(y,x)\in V\times V$ for some V in U_α. Since $y\in A$ and $(y,x)\notin A\times(X-B)$, $x\notin X-B$. (Sufficiency) Suppose there is a (x,y) in $(A\times(X-B))\cap\bigcup_{U\in\mathcal{U}_\alpha}(U\times U)$. It follows that $y\in U\subset St(x,\mathcal{U}_\alpha)$ for some U in $\mathcal{U}_\alpha$ such that $(x,y)\in U\times U$. This leads a contradiction to the hypothesis since $y\in St(x,\mathcal{U}_\alpha)\subset St(A,\mathcal{U}_\alpha)\subset B$.

REMARK: *The above result suggests an alternate definition of the associated proximity relation : $A\ll B$ if and only if $St(A,\mathcal{U}_\alpha)\subset B$ for some α in $\mathcal{A}$.*

THEOREM(2-5) *If $\Phi_1\subset\Phi_2$, then $\delta_1>\delta_2$ where $\delta_i=\delta(\Phi_i)$ for $i=1,2$.*

Proof: If $A\delta_2B$, then there is an U in $\mathcal{U}_\alpha$ such that $(A\times B)\cap(U\times U)\neq\phi$ for all $\mathcal{U}_\alpha\in\Phi_2$. But since $\Phi_1\subset\Phi_2$, for every $\mathcal{U}_\alpha\in\Phi_1$, $(A\times B)\cap(U\times U)\neq\phi$ for some $U\in\mathcal{U}_\alpha$. Thus $A\delta_1B$.

3. Completely regular family

In a completely regular topological space X, it is always true that the set $\mathcal{C}(X)$ of all continuous mappings of X into the closed unit interval is a completely regular family and the topology of X is precisely the initial topology of X and is also precisely the initial topology $\mathcal{T}(\mathcal{C})$ for the family $\mathcal{C}$. On the other hand, on every completely regular space, we can define a compatible proximity, and conversely the topology $\mathcal{T}(\delta)$ induced by a proximity is always completely regular. These argument suggest that defining a proximity by means of a suitable family of mappings is achievable.

DEFINITION: A family $\mathscr{C}$ of continuous mappings on X to the closed unit interval I is called completely regular, if for every closed set F in X and a point x not in F there is an f in $\mathscr{C}$ such that $f(x)$ is not in the closure of $f(F)$.

THEOREM(3-1) *Let $\mathscr{C}$ be a family of continuous mappings of a set X into the closed unit interval I. $\mathscr{C}$ satisfies the following condition: For each family $f_i^{-1}(F_j)$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$, where F_j's are closed in I and f_i in $\mathscr{C}$, and for every $x \notin f_i^{-1}(F_j)$ for all i,j, there exists an f_x in $\mathscr{C}$ such that $f_x(x) \notin \overline{f_x(f_i^{-1}(F_j))}$ for all i,j. Then the initial topology $\mathscr{T}(\mathscr{C})$ for the family is completely regular.*

To prove the theorem we need a lemma.

LEMMA: *Let X be a topological space and $\mathscr{C}(x)$ be a family of the continuous mappings on X into the closed unit interval such that $\mathscr{C}(x)$ is a completely regular family. Then the topological space is completely regular.*

Proof: Let x be a point not in a closed set A. Then since $\mathscr{C}(X)$ is completely regular, there is an f in $\mathscr{C}(X)$ such that $f(x) \notin \overline{f(A)}$. It implies that there is an open neighborhood $U_{f(x)}$ of $f(x)$ in I such that $U_{f(x)} \cap f(A) = \phi$. Since I is completely regular, there is an open neighborgood V of $f(x)$ such that $f(x) \in V \subset \bar{V} \subset U_{f(x)}$. Then the lemma is proved by the fact $x \in f^{-1}(V) \subset \overline{f^{-1}(V)} \subset X - A$ according to the similar step in the proof of Urysohn's Lemma.

Proof of the theorem: The family $\mathscr{C}$ becomes a set of the continuous mappings of the topological space X endowed the initial topology for the family $\mathscr{C}$ into I for which family becomes completely regular. The theorem follows from the Lemma.

THEOREM(3-2) *Let $\mathscr{C}$ be a family of mapping on a set X which satisfies the condition of theorem (3—1). Let δ be a relation on the power set of X defined by $A \bar{\delta} B$ if and only if there exists an f in $\mathscr{C}$ such that A and B are f-distinguishable. Then δ is a proximity relation on X.*

REMARK: *For the sake of convenience if $f: X \to I$, $f(A)=0$ and $f(B)=1$, then let's say A and B are f-distinguishable.*

REMARK: *From the two foregoing theorems, it is clear that $\mathscr{T}(\delta(\mathscr{C})) = \mathscr{T}(\mathscr{C})$ and $\delta(\mathscr{C}) = \delta(\mathscr{T}(\mathscr{C}))$.*

Proof of the theorem: P1, P3, and P5 are straight-forward. P2 : Suppose A and C are f-distinguishable, and B and C are g-distinguishable. Then $(A \cup B)$ and C are $f \wedge g$-distinguishable. P4: Suppose A and B are f-distinguishable. Let $E = \{x \in X : \frac{1}{2} \leq f(x) \leq 1\}$, then $A \bar{\delta} E$ and $(X-E) \bar{\delta} B$.

References

[1] S.A. Naimpally and B.D. Warrack, *Proximity spaces*, Cambridge at the University Press, 1970.
[2] J. Dugundji, *Topology*, Allyn and Bacon. Inc.. Boston, 1965.

Yonsei University