Topological ordered spaces

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L. Nachbin [8] published a note concerning topological ordered spaces and studied the relationship between topologies and ordered structures. In my previous paper [5] it is observed that every preordered structure has an equivalent topology which is called a characteristic pseudo-quasi metric topology. Consequently, every topological preordered space (X, T, P), where (X, T) is a topological space and (X, P) is a preordered space with preordered relation P is considered as a bitopological space. In this paper, the concept of characteristic pseudo-quasi metrics shall be introduced into topological preordered spaces. In particular, if the relations of interdependence between the preorder P and the topology T are given, one can obtain some meaningful results and most of Nachbin's results [8] can easily be induced as corollaries, and some of them can even be reformed (see (5.4) and (5.5)).

1. As an introduction to this paper, the following well-known defintions and theorems shall be presented in addition to a few lemmas which are used in the sequel.

(1.1) DEFINITION: A pseudo-quasi (or p.q.) metric "d" is said to be characteristic p.q. (or c.p.q.) iff the range of d is $\{0, 1\}$. [5]

For each c.p.q. metric "$\tilde{d}$" there exists the conjugate c.p.q. metric "$\underset{\sim}{d}$" which is defined as $\tilde{d}(x, y) = \underset{\sim}{d}(y, x)$.

NOTATION:

$$\tilde{S}(x, \varepsilon) = \{y : \tilde{d}(x, y) < \varepsilon, \ \varepsilon > 0\}$$
$$\underset{\sim}{S}(x, \varepsilon) = \{y : \underset{\sim}{d}(x, y) = \tilde{d}(y, x)\}$$
$$\tilde{S}(x) = \tilde{S}(x, 1), \ \ \underset{\sim}{S}(x) = \underset{\sim}{S}(x, 1)$$

(1.2) DEFINITION: Let $\tilde{C}$ be the topology the base of which is $\tilde{S}(x, \varepsilon)$. $\tilde{C}$ is said to be the *induced topology* by $\tilde{d}$. $\underset{\sim}{C}$ is also similarly defned and $(X, \tilde{C}, \underset{\sim}{C},)$ is called a *c.p.q. bitopological space*.

The separation axioms for bitopological spaces have been given suitable definitions by several authors. For instance, Kelly [3] defined "(X, T_1, T_2) is p-normal iff for each T_1-closed set A and for each T_2-closed set B disjoint from A, there is a T_1-open set V containing B and a T_2-open set U disjoint from V containing A", the terms "p-completely regular", "p-completely normal", "p-regular" and etc. have also been defined in similar manners and the following theorem has already been well-known.

(1.3) THEOREM: *Let the notation "$A \Rightarrow B$" be "A implies B",*

p.q. bitcpology $\Rightarrow$ $\left\{\begin{array}{l}\textit{p-perfectly normal} \Rightarrow \textit{p-completely normal} \Rightarrow \textit{p-normal} \\ \textit{p-completely regular} \Rightarrow \textit{p-regular}\end{array}\right.$

where "p" denotes "pairwise" (e.g. "p-regular" stand for "pairwise regular").

(1.4) THEOREM: *Let $(X, \tilde{C}, \underset{\sim}{C})$ be a c.p.q. bitopological space*

$$U \in \tilde{C} \quad iff \quad U^c \in \underset{\sim}{C}. \ [5]$$

(1.5) DEFINITON: (X, P) is a preordered space X with preorder P, which indicates the following relations "$\geq$" for all $x, y, z \in X$,

(1) $x \geq x$

Received by the editors Oct. 30, 1971

(2) $x \geq y$ and $y \geq z$ implies $x \geq z$.

(3) A preorder P is ordered iff $y \geq x$ and $x \geq y$ implies $x=y$. [8]

(1.6) NOTATION: Let P_d be the relation defined as $x \geq y$ iff $d(x,y)=0$, where "d" is a c.p.q. metric and let d_p be $d_p(x,y)=0$ iff $x \geq y$ and $d_p(x,y)=1$, otherwise, where P is a preorder. It has been shown that P_d is a preordred relation and d_p is a c.p.q. metric [5]. The following relation is obvious:

$$p_{d_{p_1}}=p_1,\ d_{p_{d_1}}=d_1, \text{ where } p_1 \text{ is a preorder and } d_1 \text{ is a c.p.q. metric.}$$

(1.7) DEFINITION: The *graph* of a preorder P on X is the subset of the square $X \times X$ formed by the points (x,y), where $x, y \in X$ such that $x \geq y$, and is denoted by G_p. [8]

2. In this chapter 2, a few properties of the characteristic pseudo-quasi bitopological space $(X, \tilde{C}, \underset{\sim}{C})$ shall be discussed.

(2.1) DEFINITION: Let (X, T, P) be a topological preordered space. P is said to be *closed* iff G_p is closed in $T \times T$.

(2.2) DEFINITION: Let (X, T, P) be a topological preordered space. $A \subset X$ is said to be decreasing, if $a \leq b$ and $b \in A$ imply $a \in A$. $B \subset X$ is said to be increasing, if $a \geq b$ and $b \in B$ imply $a \in B$.

NOTATION:
$$D(A)=\{y : \tilde{d}_p(x,y)=0 \text{ or } x \geq y,\ x \in A\}.$$
$$I(A)=\{y : \underset{\sim}{d}_p(x,y)=0=\tilde{d}_p(y,x) \text{ or } x \leq y,\ x \in A\}.$$

REMARK(i) Consider $(X, \tilde{C}, \underset{\sim}{C})$, where $\tilde{C}$ is the topology which is generated by $\tilde{d}_p$. If A is decreasing; $A=\{b : a \geq b,\ a \in A\}=\{b : \tilde{d}_p(a,b)=0,\ a \in A\}=\bigcup_{a \in A} \tilde{S}(a)$. Therefore $A \in \tilde{C}$. Similarly, if B is increasing, $B \in \underset{\sim}{C}$.

REMARK(ii) It is easy to show the following: $D(A)=\bigcup_{x \in A} \tilde{S}(x)$, and $D(A)$ is the smallest decreasing set containing A, or the minimum nbhd of A with respect to $\tilde{C}$. Similarly, $I(A)$ is the smallest incresing set containing A, or the minimum nbhd of B with respect to $\underset{\sim}{C}$.

(2.3) DEFINITION: $A \subset X$ is convex iff $a \leq b \leq c$ and $a, c \in A$ imply $b \in A$, and (X, T) is locally convex iff the set of convex nbhds is a base for T.

$C(A)=\bigcup_{a \in A}(\tilde{S}(a) \cap \underset{\sim}{S}(a))$ is the smallest convex set containing A in $\{\tilde{C} \cup \underset{\sim}{C}\}$. Since $(X, \tilde{C}, \underset{\sim}{C})$ is a p.q. bitopological space, it enjoys many properties of bitopological spaces (see (1.3)). In particular, every $\tilde{C}$-open set is a decreasing set, every $\underset{\sim}{C}$-open set is an increasing set. Furthermore, we have

(2.4) LEMMA: *Let $(X, \tilde{C}, \underset{\sim}{C})$ be a c.p.q. bitopological space. If $f \in ULX$ (f is $\tilde{C}$-upper semi-continuous and $\underset{\sim}{C}$-lower semi-continuous), then f is a both $P_{\tilde{d}}$ and $P_{\underset{\sim}{d}}$-increasing function, where $\tilde{d}$ is the c.p.q. metric which generates $\tilde{C}$.*

Proof. Let f be $\tilde{C}$-upper semi-continuous. Therefore, $\{y : f(y) \geq a\}$ is a $\tilde{C}$-closed subset. Consequently, it is $\underset{\sim}{C}$-open by (1.4), for all $a \in R$. Then $\underset{\sim}{S}(x) \subset \{z : f(z) \geq f(x)=a\} \in \underset{\sim}{C}$. Therefore, if $y \geq x$, then $f(y) \geq f(x)$. On the other hand, consider $\{y : f(y) \leq a\}$. Since f is $\underset{\sim}{C}$-lower semi-continuous, it is $\underset{\sim}{C}$-closed, and consequetly $\tilde{C}$-open and

$$\tilde{S}(X) \subset \{z : f(z) \leq f(x)=a\} \in \tilde{C},$$

i.e. $y \leq x$ implies $f(y) \leq f(x)$ and the proof is completed.

Kelly [3] proved the following theorem:

(2.5) THEOREM: *If (X, T_1, T_2) is P-normal, then given a T_2-closed set F and a T_1-closed set H with $F\cap H=\phi$, there exists $f\in ULX$ such that $f(x)=0(x\in F)$, $f(x)=1$ $(x\in H)$, and $0\leq f(x)\leq 1$ $(x\in X)$.*

3. In this chapther, we shall discuss the relationship between the topology and the c.p.q. topology for a given topological preordered space (X, T, P).

(3.1) LEMMA: *Let (X, T, P) be a topological preordered space and*

$$G=\{(x,y) : d_p(x,y)=0 \text{ or } x\geq y\}.$$

If G is closed in $T\times T$, then $G[x]=\{y : (x,y)\in G\}$ is closed in T for all $x\in X$.

Proof: Assume that there exists $x\in X$ such that $G[x]$ is not closed in T. Let q be a limit point of $G[x]$, and $q\notin G[x]$ which implies $(x,q)\notin G$. Let V and W be any open sets in T, which contain x and q, respectively. Since $W\cap G[x]\neq\phi$, $(V\times W)\cap G\neq\phi$ and (x,q) is a limit point of G but $(x,q)\notin G$, which is a contradiction. Therefore $G[x]$ is closed.

(3.2) LEMMA: *Let (X, T, P) be a topological preordered space.*

$G=\{(x,y) : d_p(x,y)=0\}$ is closed iff $G^{-1}=\{(y,x) : (x,y)\in G\}$ is closed.

Proof: Let $q=(a,b)\in X\times X$ be a limit point of G^{-1}. Then $(W_q-q)\cap G^{-1}\neq\phi$, where $W_q=W_a\times W_b$, $W_a, W_b\in T$ with $a\in W_a$ and $b\in W_b$. Therefore, there exists (a',b') in $(W_a\times W_b)\cap G^{-1}$, where $(a,b)\neq(a',b')$. Equivalently, $(b',a')\in(W_b\times W_a)-(b,a)$, which implies (b,a) is a limit point of G. If G is closed, $(b,a)\in G$ and $(a,b)\in G^{-1}$; consequently, G^{-1} is closed. The converse is similarly obtained.

(3.3) THEOREM: *Let (X, T, P) be a topological preordered space and $(X, \tilde{C}, \underset{\sim}{C})$ be the c.p.q. bitopological space, where $\tilde{C}$ is generated by d_p, G the graph of P is closed in $T\times T$ iff $\tilde{C}\cup\underset{\sim}{C}\subset T$.*

Proof: If G is closed in $T\times T$, then $G[x]=\{y:(x,y)\in G\}=\tilde{S}(x)$ is closed in T for all $x\in X$ by (3.1) and $\underset{\sim}{C}\subset T$. By (3.2) G is closed iff G^{-1} is closed, where $G^{-1}=\{(x,y) : (y,x)\in G\}$ and $G^{-1}[x]=\underset{\sim}{S}(x)$. By the same discussion $\tilde{C}\subset T$. To show the converse, assume $T\supset\tilde{C}\cup\underset{\sim}{C}$. Let $t=(x_1,y_1)\notin G$. Assume $(x_2,y_2)\in(\tilde{S}(x_1)\times\underset{\sim}{S}(y_1))\cap G$ and $(x_2,y_2)\neq\phi$. Since $\tilde{d}_p$ is a p.q. metric $\tilde{d}_p(x_1,y_1)\leq\tilde{d}_p(x_1,x_2)+\tilde{d}_p(x_2,y_2)+\tilde{d}_p(y_2,y_1)$. However, $\tilde{d}_p(x_1,y_1)=1$, because $(x_1,y_1)\notin G$, $\tilde{d}_p(x_2,y_2)=0$, since $(x_2,y_2)\in G$, furthermore, $\tilde{d}_p(x_1,x_2)=0$ and $\underset{\sim}{d}_p(y_1,y_2)=\tilde{d}_p(y_2,y_1)=0$, which is a contradiction, therefore $(x_2,y_2)=\phi$ and G is closed in $T\times T$.

Now let us consider a non-trivial example, where T is non-discrete, both $\underset{\sim}{C}$ and $\tilde{C}$ are not indiscrete, and $\tilde{C}$, $\underset{\sim}{C}\subset T$.

(3.4) EXAMPLE: Let X be reals and $\tilde{d}$ be defined as $\tilde{d}(x,y)=0$ if $1+[|1|]\geq y$, 1 if $1+[|1|]<y$, where $[|x|]$ is the greatest integer function. Let $\underset{\sim}{d}$ be the conjugate of $\tilde{d}$. Then $\tilde{C}$ has the base $\{(-\infty,n]\}$, n is an integer, while, $\underset{\sim}{C}$ has the base $\{(n,\infty)\}$, and let T be generated by $\{(n,n+1]\}$.

REMARK: $(X, C_p)\supset(X, C_{p'})$ iff $G_p\subset G_{p'}$, (the graph of $C_{p'}$, the c.p.q. topology generated by the preorder P'), iff the identity funtion I: $(X, C_p)\to(X, C_{p'})$ is continuous, or equivalently $I:(X,P)\to(X,P')$ is an increasing function. The followings have been proved by Nachbin (see p.26 [8]). Here their proofs shall be presented in a straightforward manner in terms of $(X, \tilde{C}, \underset{\sim}{C})$ the c.p.q. bitopology.

(3.5) THEOREM: *Let (X,T,P) be a topological preordered space. P is closed iff for every $(a,b)\notin G$ (the graph of P) there exist V such that $b\in V\in \underset{\sim}{C}$ (or increasing nbhd of b) and W such that $a\in W\subset \tilde{C}$ (or decreasing nbhd of a) with $V\cap W=\phi$.*

Proof: Since P is closed $\tilde{C}, \underset{\sim}{C}\subset T$, by(3.3). $(a,b)\notin G$ implies $\tilde{d}_p(a,b)=1$ and $\underset{\sim}{d}(b,a)=1$. $\tilde{S}(a)=\{y:\tilde{d}_p(a,y)=0\}\in\tilde{C}$, $\underset{\sim}{S}(b)=\{y:\underset{\sim}{d}_p(b,y)=0\}\in C$. If $z\in\tilde{S}(a)\cap\underset{\sim}{S}(b)$, $\tilde{d}_p(a,z)=0=\underset{\sim}{d}_p(b,z)$ but $\tilde{d}_p(a,b)\leq\tilde{d}_p(a,z)+\tilde{d}_p(z,b)=\tilde{d}_p(a,z)+\underset{\sim}{d}_p(b,z)$. It is a contradiction, therfore, $\tilde{S}(a)\cap\underset{\sim}{S}(b)=\phi$. Recall that an element of $\tilde{C}$ and $\underset{\sim}{C}$ is a decreasing and an increasing set (see the remark, (2.2)), respectively. The converse is trivial and the proof is completed.

(3.6) THEOREM: *Let (X,T,P) be a topological preordered space. If P is closed, then $I(a)$ and $D(a)$ are closed for all $a\in X$.*

Proof: Since P is closed, $\tilde{C}, \underset{\sim}{C}\subset T$ and $I(a)=\underset{\sim}{S}(a)$, $D(a)=\tilde{S}(a)\in\tilde{C}\cup\underset{\sim}{C}\subset T$, and by applying(1.4), the proof will be completed. In the process of the proof of the above theorem (3.6), one can also see the fact that $I(a)$ and $D(a)$ are both open and closed in T and $I(a)\cap D(a)\in T$. Therefore, we have the following theorem:

(3.7) THEOREM: *Let (X,T,P) be a topological preordered space. If P is closed, then $I(a)$ and $D(a)$ are both open and closed.*

The following is easy to prove:

(3.8) THEOREM: *$\tilde{C}\cup\underset{\sim}{C}=T$ iff T is locally convex.*

(3.9) LEMMA: *Let $(X,\tilde{C},\underset{\sim}{C})$ be a c.p.q. bitopological space. If $P_{\tilde{d}}$ (the preorder of $\tilde{C}$) is orderd, then $(X,\tilde{C},\underset{\sim}{C})$ is $P\text{-}T_{1\frac{1}{2}}$) (refer [4] for$P\text{-}T_{1\frac{1}{2}}$)*

Proof: Since $P_{\tilde{d}}$ is ordered, $a\neq b$ implies either one of the $a\geq b$ and $b\geq a$ is false. Without loss of generality, assume $a\geq b$ is false. If $\phi\neq c\in\tilde{S}(a)\cup\underset{\sim}{S}(b)$, then $\tilde{d}(a,b)=1$, $\tilde{d}(a,c)=0$ and $\underset{\sim}{d}(b,c)=\tilde{d}(c,b)=0$ which is a contradiction of the triangular inequality of $\tilde{d}$. Consequently, $\tilde{S}(a)\cap\underset{\sim}{S}(b)=\phi$.

(3.10) COROLLARY: *Let (X,T,P) be a topological ordered space. If P is closed, then it is Hausdorff.*

4. In this chapter, we shall discuss normally preordered spaces.

If $\tilde{C}$ is the c.p.q. topology which is generated by $d_{\tilde{p}}$, where P is the preorder of a given topological ordered space (X,T,P), then we have the following: $A\in\underset{\sim}{C}\cap T$ iff A^c is closed in $\underset{\sim}{C}$ and T, which is equivalent to say A^c is a decreasing and closed set in (X,T,P). Similarly, $B\in\tilde{C}\cap T$ iff B^c is increasing and closed. Therefore, the following Nachbin's definition (p.28[8]) has a version in terms of bitopological spaces.

(4.1) DEFINITION (Nachbin): (X,T,P) is a normally preordered space iff for every two closed subsets F_0 and F_1 such that $F_0\cap F_1=\phi$, where F_0 is decreasing and F_1 is increasing, there exist an increasing open set A_0 and a decreasing open set A_1 such that $F_1\subset A_1$, $F_0\subset A_0$ and $A_0\cap A_1=\phi$.

With the previous remark and the fact that $A\in\tilde{C}\cup T$ iff A is a decreasing open set (see the remark(i) of (2.2)), the above definition is restated as follows:

(4.2) DEFINITION: (X,T,P) is a normally preordered space iff for every $T\cap\tilde{C}$-closed set F_0 and $T\cap \underset{\sim}{C}$-closed set F_1 where $F_0\cap F_1=\phi$, there exist $A_0\in T\cap \underset{\sim}{C}$ and $A_1\in T\cap \tilde{C}$ such that $F_0\subset A_0$ and $F_1\subset A_1$ with $A_0\cap A_1=\phi$ or simply, $(X, T\cap\tilde{C}, T\cap\underset{\sim}{C})$ is p-normal.

By considering (2.4) and (2.5), one can easily obtain the following which has been proved by Nachbin (p.30, [8]) for the first time, as corollary to the theorem (2.5):

(4.3) COROLLARY: *(X,T,P) is normally preordered iff for any disjoint closed sets F_0 and F_1 where F_0 is decreasing and F_1 is increasing, there exists a continuous increasing function f such that $f(F_0)=0$, $f(F_1)=1$ and $0\leq f(x)\leq 1$.*

Proof: By (2.5), there exists $f\in ULX$ in $(X, \tilde{C}\cap T, \underset{\sim}{C}\cap T)$, with the properties contained in the theorem (2.5). Since $T\supset(\tilde{C}\cap T)\cup(\underset{\sim}{C}\cap T)$, f is continuous. Moreover, by (2.4), f is also an increasing function.

By the same method, the generalized Tieze's extension theorem which has been shown by Lane [6] will be applied to obtain a similar result. However, the reult is omitted.

Now let us consider compact ordered spaces, which have been defined by Nachbin as in the following:

(4.4) DEFINITION: (X, T, P) is a compact ordered space iff (X, T) is compact and G_p is closed.

(4.5) LEMMA: *If (X, T, P) is a compact ordered space, then it is a Hausdorff space.*

Proof: See (3.10).

It is easy to see $T\supset\tilde{C}\cup\underset{\sim}{C}$, if (X, T, P) is a compact ordered space. In fact, the following shows that $T=\tilde{C}\cup\underset{\sim}{C}$.

(4.6) THEOREM: *If (X, T, P) is a compact ordered space, then $T=\tilde{C}\cup\underset{\sim}{C}$.*

Proof: Since G_p is closed, $T\supset\tilde{C}\cup\underset{\sim}{C}$. Let K be a closed subset in T. The compactness of (X, T) implies that any closed subset $K\subset X$ is a compact subset. By (3.9) $(X, \tilde{C}, \underset{\sim}{C})$ is $P\text{-}T_{1\frac{1}{2}}$. Therefore, for each $b\in K$, and $a\notin K$, there exist either disjoint $\tilde{S}(a)$ and $\underset{\sim}{S}(b)$ or disjoint $\underset{\sim}{S}(a)$ and $\tilde{S}(b)$. The compactness of K implies that K is closed in $\{\tilde{C}\cup\underset{\sim}{C}\}$ and that $\tilde{C}\cup\underset{\sim}{C}\supset T$.

From the fact that $(X, \tilde{C}, \underset{\sim}{C})$ is $P\text{-}T_{1\frac{1}{2}}$ and the compactness of (X, T) and by using a method similar to the above proof, we can easily obtain the following corollary:

(4.7) COROLLARY: *Let (X, T, P) be a compact ordered space. If $F_0, F_1\subset X$ are closed sets such that $x_0\geq x_1$ is false for any $x_0\in F_0$ and $x_1\in F_1$, then there exist a decreasing open set V_0 and an increasing open set V_1 such that $F_0\subset V_0$, $F_1\subset V_1$ and $V_0\cap V_1=\phi$.*

(4.8) COROLLARY: *every compact ordered space is normally ordered.*

Proof: See (1.3) and (4.6).

5.

(5.1) DEFINITION(Nachbin[8]): (X, T, P) is uniformizable iff (1) for $a\in X$ and $V\in T$ such that $a\in V$, there exist f, g, where f is increasing and g is decreasing with $0\leq f(x)\leq 1$, $0\leq g(x)\leq 1$, $f(a)=1=g(a)$, and $inf(f(x),\ g(x))=0$ *if* $x\in V$: (2) if $a,\ b\in X$ and $a\leq b$ is false, then there exists a continuous increasing function f on X such that $f(a)>f(b)$.

With the above definition, he showed that such a uniformizable space has as a subbase a set of all decreasing and increasing sets (Proposition 6. p. 53, [8]). What he states is "Let (X, T, P) be a uniformizable preordered space. Then the preorder is closed, and the set formed by open decreasing and the open increasing subset is an open subbase". The theorem is essentially the same as saying $(X, \{\tilde{C} \cup \underset{\sim}{C}\}, P) = (X, T, P)$, whenever the conditions are given. Now recall (1.3) and the definition of P-completely regular. Then the following is trivial:

(5.2) COROLLARY: *The topology of every uniform preordered space is locally convex.*

(5.3) COROLLARY: *Every compact ordered space is a uniform ordered space* See (4.6) for the proof.

The following are reformed theorems of Nachbin's results (Proposition 12, p.76 and propostion 14, p. 84 [8])

(5.4) THEOREM: *A topological Abelian group X, with preorder P is a uniform preordered space iff for every nbhd A of the element 0, there exists another nbhd B such that $0 \leq x \leq y \in B$ implies $x \in A$.*

Proof: Since the condition is equivalent to say that X is locally convex, by (3.8) and the remark of (5.1), the result is trivial.

Nachbin had another condition beside the above that the set M of the positive elements of X be closed. However, refering to (3.8) and (3.3) it can be deleted. By the same reason, one obtains the next theorem deleting one condition from the Nachbin's theorem.

(5.5) THEOREM: *Let X be a locally convex vector space (in the usual vector sense) with preorder P. The nbhd of every point which is convex both in the sense of preorder and in the vector sense forms a base iff for any nbhd A of 0 there exists a nbhd B such that $0 \leq x \leq y \in B$ implies $x \in A$.*

Proof: Let $U \subset X$ be a convex subset in the vector sense and

$$C(U) = \{y : y \in \tilde{S}(x_1) \cap \underset{\sim}{S}(x_2),\ x_i \in U\}.$$

We need the following 3 steps to complete the proof.

(i) $C(U)$ is a convex set in the vector sense.

For, let $a, b \in C(U)$ and $c = ta + (1-t)b$, $0 \leq t \leq 1$. Then, $a \in \tilde{S}(a_1) \cap \underset{\sim}{S}(a_2)$ (or equivalently $a_2 \leq a \leq a_1$), and $b \in \tilde{S}(b_1) \cap \underset{\sim}{S}(b_2)$, where $a_1, a_2, b_1, b_2 \in U$. Consider $c_1 = ta_1 + (1-t)b_1$ and $c_2 = ta_2 + (1-t)b_2$. Then $c_1, c_2 \in U$, because U is a convex set in the vector sense and $c_1 \geq c \geq c_2$ implies $c \in \tilde{S}(c_1) \cap \underset{\sim}{S}(c_2) \in C(U)$.

(ii) The nbhds of every point which is convex in the sense of preorder form a base (see (5.4)).

(iii) Let A be an arbitrary nbhd of 0. By (ii) $\tilde{S}(0) \cap \underset{\sim}{S}(0) \subset A$. Since X is convex in the vector sense, there exists a vector convex nbhd K such that

$$0 \in K \subset \tilde{S}(0) \cap \underset{\sim}{S}(0) \in A.$$

Consider $C(K) = \bigcup_{a_i \in k} \tilde{S}(a_1) \cap \underset{\sim}{S}(a_2) = \tilde{S}(0) \cap \underset{\sim}{S}(0) \subset A$. By (i), $C(K)$ is convex in the vector sense and the proof is completed.

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