A certain submanifold of codimension 2 of a Kählerian manifold

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§ 0. Introduction.

It is well known that a submanifold of codimension 2 of an almost Hermitian manifold or a hypersurface of an almost contact metric manifold carries an (f, g, u, v, λ)-structure ([1], [9]).

Let E be an even-dimensional Euclidean space, it can be regarded as a flat Kählerian manifold. The submanifolds M of E carrying such a structure with certain conditions have studied in [5], [7], [9].

In [5], [9], K. Yano and M. Okumura proved:

THEOREM A. *Let M be a complete differentiable submanifold of codimension 2 of an even-dimensional Euclidean space and C and D be two normals to M. (1) If the connection induced in the normal bundle of M is trivial and the (f, g, u, v, λ)-structure induced on M is normal or (2) $\lambda \neq$ constant, H and K commute with f, then M is a plane, a sphere or a product of a sphere and a plane, where H and K are the Weingarten maps with respect to C and D respectively.*

In the present paper, we study a submanifold M of codimension 2 of a Kählerian manifold $\tilde{M}$ whose tangent space is invariant under the curvature transformation of $\tilde{M}$.

In section 1, we recall the properties of submanifold of codimension 2 in a Kählerian manifold.

In section 2, we find differential equations the induced (f, g, u, v, λ)-structure satisfies and some relations under the condition (2) in Theorem A.

In the last section 3, we give a complete classification of the submanifold.

In the sequel we assume that $\lambda \neq$ constant on the submanifold M. Our main result appears in Theorem 3. 2.

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§ 1. Submanifolds of codimension 2 of a Kählerian manifold.

Let $\tilde{M}$ be a $(2n+2)$-dimensional Kählerian manifold covered by a system of coordinate neighborhoods $\{\tilde{U}; y^\kappa\}$, where here and in the sequel the indices $\kappa, \lambda, \mu, \nu, \cdots\cdots$ run over the range $\{1, 2, \ldots\ldots, 2n+2\}$ and let $(F_\lambda{}^\kappa, G_{\mu\lambda})$ be the Kählerian structure, that is

(1. 1) $$F_\mu{}^\kappa F_\lambda{}^\mu = -\delta_\lambda{}^\kappa,$$

and $G_{\mu\lambda}$ a Riemannian metric such that

(1. 2) $$G_{\gamma\beta} F_\mu{}^\gamma F_\lambda{}^\beta = G_{\mu\lambda},$$

and

(1. 3) $$\nabla_\mu F_\lambda{}^\kappa = 0,$$

where we denote by $\{{}_\mu{}^\kappa{}_\lambda\}$ and ∇_μ the Christoffel symbols formed with $G_{\mu\lambda}$ and the opera-

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tor of covariant differentiation with respect to $\{{}_\mu{}^\kappa{}_\lambda\}$ respectively.

Let M be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U;\ x^h\}$, where here and in the sequel the indices $h, i, j, \ldots\ldots$ run over the range $\{1, 2, \ldots\ldots, 2n\}$ and which is differentiably immersed in $\tilde{M}$ as a submanifold of codimension 2 by the equations

(1.4) $$y^\kappa = y^\kappa(x^h).$$

We put

(1.5) $$B_i{}^\kappa = \partial_i y^\kappa, \qquad (\partial_i = \partial/\partial x^i),$$

then $B_i{}^\kappa$ is, for each i, a local vector field of $\tilde{M}$ tangent to M and the vectors $B_i{}^\kappa$ are linearly independent in each coordinate neighborhood. $B_i{}^\kappa$ is, for each κ, a local 1-form of M.

We choose two mutually orthogonal unit vectors C^κ and D^κ of $\tilde{M}$ normal to M in such a way that $2n+2$ vectors $B_i{}^\kappa$, C^κ, D^κ give the positive orientation of $\tilde{M}$.

The transforms $F_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $F_\lambda{}^\kappa$ can be expressed

(1.6) $$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa + v_i D^\kappa,$$

where $f_i{}^h$ is a tensor field of type (1, 1) and u_i, v_i are 1-forms of M. Similarly the transform $F_\lambda{}^\kappa C^\lambda$ of C^λ by $F_\lambda{}^\kappa$ and the transform $F_\lambda{}^\kappa D^\lambda$ of D^λ by $F_\lambda{}^\kappa$ can be written as

(1.7) $$F_\lambda{}^\kappa C^\lambda = -u^i B_i{}^\kappa + \lambda D^\kappa,$$
$$F_\lambda{}^\kappa D^\lambda = -v^i B_i{}^\kappa - \lambda C^\kappa,$$

where

$$u^i = u_t g^{ti}, \quad v^i = v_t g^{ti},$$

g_{ji} being the Riemannian metric on M induced from that of $\tilde{M}$, and λ is a function on M. We can easily verify that λ is a function globally defined on M.

From (1.2), (1.6) and (1.7) we have ([8])

$$f_j{}^t f_t{}^h = -\delta_j{}^h + u_j u^h + v_j v^h,$$
$$f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i,$$
(1.8) $$f_i{}^t u_t = \lambda v_i \quad \text{or} \quad f_i{}^h u^i = -\lambda v^h,$$
$$f_i{}^t v_t = -\lambda u_i \quad \text{or} \quad f_i{}^h v^i = \lambda u^h,$$
$$u_i u^i = v_i v^i = 1 - \lambda^2, \qquad u_i v^i = 0.$$

If we put

$$f_{ji} = f_j{}^r g_{ri},$$

then we can easily verify that f_{ji} is skew-symmetric.

§2. Structure equations of submanifold of codimension 2.

We denote by $\{{}_j{}^h{}_i\}$ and ∇_i the Christoffel symbols formed with g_{ji} and the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$ respectively. Then the equations of Gauss and Weingarten are respectively

(2.1) $$\nabla_j B_i{}^\kappa = \partial_j B_i{}^\kappa + \{{}_\mu{}^\kappa{}_\lambda\} B_j{}^\mu B_i{}^\lambda - B_h{}^\kappa \{{}_j{}^h{}_i\} = h_{ji} C^\kappa + k_{ji} D^\kappa,$$

(2.2) $$\nabla_j C^\kappa = \partial_j C^\kappa + \{{}_\mu{}^\kappa{}_\lambda\} B_j{}^\mu C^\lambda = -h_j{}^i B_i{}^\kappa + l_j D^\kappa,$$
$$\nabla_j D^\kappa = \partial_j D^\kappa + \{{}_\mu{}^\kappa{}_\lambda\} B_j{}^\mu D^\lambda = -k_j{}^i B_i{}^\kappa - l_j C^\kappa,$$

where h_{ji} and k_{ji} are the second fundamental tensors with respect to the normals C^κ and D^κ respectively, $h_j{}^h$ and $k_j{}^h$ define the corresponding Weingarten maps H and K and are given by

$$h_j{}^h = h_{ji} g^{ih}, \qquad k_j{}^h = k_{ji} g^{ih},$$

l_j is the third fundamental tensor.

Differentiating (1.6) covariantly along M and taking account of (1.3), (2.1) and (2.2), we get ([2])

(2.3) $$\nabla_j f_i{}^h = -h_{ji} u^h + h_j{}^h u_i - k_{ji} v^h + k_j{}^h v_i,$$

(2.4) $$\nabla_j u_i = -h_{ji} f_i{}^t - \lambda k_{ji} + l_j v_i,$$

(2.5) $$\nabla_j v_i = -k_{ji} f_i{}^t + \lambda h_{ji} - l_j u_i.$$

Similarly, from (1.7) we find

(2.6) $$\nabla_j \lambda = k_{ji} u^i - h_{ji} v^i.$$

Now, we consider a submanifold M of codimension 2 of a Kählerian manifold satisfying the following conditions:

(2.7) $$f_j{}^t h_t{}^h = h_j{}^t f_t{}^h \quad \text{and} \quad f_j{}^t k_t{}^h = k_j{}^t f_t{}^h.$$

From (2.7), we can find

(2.8) $$h_{ji} u^i = \alpha u_j, \qquad h_{ji} v^i = \alpha v_j,$$

(2.9) $$k_{ji} u^i = \bar{\alpha} u_j, \qquad k_{ji} v^i = \bar{\alpha} v_j,$$

and

(2.10) $$\bar{\alpha} h_{ji} = \alpha k_{ji},$$

where α and $\bar{\alpha}$ are functions on M [2].

We here assume that, for a submanifold M of codimension 2 of a Kählerian manifold $\tilde{M}$, all curvature transformations preserve the tangent space $T_p(x(M))$ at each point p of $x(M)$, where x is an immersion $M \to \tilde{M}$, that is, the tangent space of M is invariant under the curvature transformation of $\tilde{M}$ [10].

Then the equations of Codazzi are

(2.11) $$\nabla_k h_{ji} - \nabla_j h_{ki} = l_k k_{ji} - l_j k_{ki},$$

(2.12) $$\nabla_k k_{ji} - \nabla_j k_{ki} = l_j h_{ki} - l_k h_{ji}.$$

§ 3. Submanifolds of codimension 2 of a Kählerian manifold such that the curvature transformation is invariant.

In this section, we assume that, for a submanifold M of codimension 2 of a Kählerian manifold $\tilde{M}$, (1) the tangent space of M is invariant under the curvature transformation of $\tilde{M}$ and (2) the conditions (2.7) are satisfied.

Differentiating the first equation of (2.8) covariantly, we find

$$(\nabla_k h_{ji}) u^i + h_{ji} \nabla_k u^i = (\nabla_k \alpha) u_j + \alpha \nabla_k u_j,$$

from which, using (2.4) and (2.11)

$$(l_k k_{ji} - l_j k_{ki}) u^i + h_{ji}(-h_{kt} f^{ti} - \lambda k_k{}^i + l_k v^i) - h_{ki}(-h_{jt} f^{ti} - \lambda k_j{}^i + l_j v^i)$$
$$= (\nabla_k \alpha) u_j - (\nabla_j \alpha) u_k + \alpha(-h_{kt} f_j{}^t + h_{jt} f_k{}^t + l_k v_j - l_j v_k),$$

or, using (2.7), (2.8) and (2.9)

(3.1) $$-2h_{ji} h_{kt} f^{it} - \lambda(h_{ji} k_k{}^i - h_{ki} k_j{}^i) = (\nabla_k \alpha - \bar{\alpha} l_k) u_j - (\nabla_j \alpha - \bar{\alpha} l_j) u_k - 2\alpha h_{kt} f_j{}^t.$$

Transvecting (3.1) with u^j, we find

$$\nabla_k \alpha - \bar{\alpha} l_k = \mu u_k,$$

for a function μ.

Starting from the second equation of (2.8), we have

$$\nabla_k\alpha-\bar{\alpha}l_k=\mu' v_k.$$

From the last two equations, we have

$$\nabla_k\alpha=\bar{\alpha}l_k \tag{3.2}$$

by virtue of the orthogonality of u_k and v_k.

Substituting (3.2) into (3.1) and using (2.7), we find

$$2(h_{ki}h_t{}^i-\alpha h_{kt})f_j{}^t=\lambda(h_{ki}k_j{}^i-h_{ji}k_k{}^i),$$

from which, transvecting $f_h{}^j$ and using (1.8) and (2.8),

$$2(h_{ki}h_h{}^i-\alpha h_{kh})=\lambda(h_{ki}k_t{}^i-h_{ti}k_k{}^i)f_h{}^t. \tag{3.3}$$

In the same way, we can find from (2.9)

$$\nabla_k\bar{\alpha}=-\alpha l_k \tag{3.4}$$

and

$$2(k_{ki}k_h{}^i-\bar{\alpha}h_{kh})=\lambda(h_{ki}k_t{}^i-h_{ti}k_k{}^i)f_h{}^t. \tag{3.5}$$

We now prove

LEMMA 3.1 *Let M be a submanifold of codimension 2 of a Kählerian manifold $\tilde{M}$ whose tangent space is invariant under the curvature transformation of $\tilde{M}$. If the conditions (2.7) are satisfied, then*

(1) *the connection of the normal bundle of M is trivial,*

(2) *the mean curvature of M is constant,*

and consequently

(3) *the mean curvature vector of M does not vanish everywhere on M.*

Proof. Differentiating (3.4) covariantly and substituting (3.2) into the resulting equation, we find

$$\nabla_k\nabla_j\bar{\alpha}=-\bar{\alpha}l_kl_j-\alpha\nabla_kl_j,$$

from which

$$\alpha(\nabla_kl_j-\nabla_jl_k)=0.$$

In the same way,

$$\bar{\alpha}(\nabla_kl_j-\nabla_jl_k)=0.$$

Since $\nabla_j\lambda=\bar{\alpha}u_j-\alpha v_j$ and $\lambda\neq$ const., α and $\bar{\alpha}$ are not always zero.

Thus the connection of the normal bundle of M is trivial. So, we can choose the unit normals C^κ and D^κ in such a way that $l_j=0$. In this case α and $\bar{\alpha}$ are both constants because of (3.2) and (3.4).

From (2.10) and $(\alpha,\bar{\alpha})\neq(0,0)$, we see that

$$h_{ki}k_j{}^i=k_{ki}h_j{}^i.$$

Therefore, (3.3) and (3.5) can be respectively written as

$$h_{ji}h_k{}^i=\alpha h_{jk}, \tag{3.6}$$

and

$$k_{ji}k_k{}^i=\bar{\alpha}k_{jk}. \tag{3.7}$$

(3.6) shows that $h_j{}^i$ has only two constant eigenvalues α and 0. Similarly $k_j{}^i$ has $\bar{\alpha}$ and 0. Now, let r and s be multiplicities of α of $h_j{}^i$ and of $\bar{\alpha}$ of $k_j{}^i$ respectively, then

$$h_i{}^i=r\alpha=\text{const.},\qquad k_i{}^i=s\bar{\alpha}=\text{const.}.$$

Substituting these into the equation giving the mean curvature of the submanifold M;

$$H^2=\frac{1}{4n^2}\{(h_i{}^i)^2+(k_i{}^i)^2\},$$

we have $H=$const..

Suppose that the submanifold M is minimal.

Then, from (3.6) and (3.7) we have

$$h_{ji}h^{ji}=0, \qquad k_{ji}k^{ji}=0,$$

from which

$$h_{ji}=0, \qquad k_{ji}=0.$$

Thus, the equation (2.6) shows that λ is constant. This contradicts to the assumption. From the equation giving the mean curvature vector of M;

$$H^\kappa=\frac{1}{2n}(h_i{}^iC^\kappa+k_i{}^iD^\kappa), \tag{3.8}$$

we see that H^κ does not vanish everywhere on M. This completes the proof of the lemma.

From $H^\kappa\neq 0$ on M, we can choose the first unit normal C^κ along the direction of H^κ and choose the second unit normal D^κ in such a way that $B_i{}^\kappa$, C^κ, D^κ form the positive orientation of the Kählerian manifold $\tilde{M}$. Then, from (3.8), we see that

$$k_i{}^i=0,$$

which implies that

$$k_{ji}k^{ji}=0,$$

because of (3.7). Thus we have

$$k_{ji}=0, \tag{3.9}$$

from which, using (2.12),

$$h_t{}^il^t=h_t{}^tl^i,$$

which shows that l^i is an eigenvector of $h_j{}^i$ corresponding to the eigenvalue $h_t{}^t$. So we have

$$h_i{}^i=0 \quad \text{or} \quad h_i{}^i=\alpha.$$

The submanifold M not being minimal, we have $h_i{}^i=\alpha$. This implies that l^i vanishes identically on M because of (2.8).

Thus the equations (2.1) and (2.2) can be respectively written as

$$\nabla_jB_i{}^\kappa=h_{ji}C^\kappa, \tag{3.10}$$

and

$$\nabla_jC^\kappa=-h_j{}^iB_i{}^\kappa, \qquad \nabla_jD^\kappa=0. \tag{3.11}$$

We here prove the following

THEOREM 3.2. *Let M be a complete connected submanifold of codimension 2 of a Kählerian manifold $\tilde{M}$ whose tangent space is invariant under the curvature transformation of $\tilde{M}$. If H and K commute with f and $\lambda\neq$const., then M is a sphere or a product of a sphere and a totally geodesic submanifold, when H and K are defined in section 2.*

Proof. M not being minimal, we can see that from (3.6) M is totally umbilical (i.e. $h_{ji}=\alpha g_{ji}$, $\alpha\neq 0$) or $h_j{}^i$ has exactly two distinct constant eigenvalues α and 0, where $\alpha\neq 0$

has multiplicity r.

In the first case M is a sphere S^{2n}, in fact, (2.5) and (2.6) become

(3.12) $$\nabla_j v_i = \lambda h_{ji} = \alpha\lambda g_{ji},$$

(3.13) $$\nabla_j \lambda = -\alpha v_j$$

by virtue of (3.9) and $l_j = 0$.

Differentiating (3.13) covariantly and using (3.12), we have

$$\nabla_j \nabla_i \lambda = -\alpha^2 \lambda g_{ji}, \quad \alpha = \text{const.}.$$

This implies that the submanifold M is isometric to S^{2n} [4].

In the second case, the correspondences a point $P \in M$ to

$$D_\alpha(P) = \{w^i \in T(M) \mid h_j{}^i w^j = \alpha w^i\},$$

and to

$$D_0(P) = \{w^i \in T(M) \mid h_j{}^i w^j = 0\},$$

define two mutually orthogonal distributions of dimension r and $2n-r$ respectively. Since α is constant, by virtue of Codazzi-equation;

(3.14) $$\nabla_k h_{ji} - \nabla_j h_{ki} = 0,$$

we know that both distributions are integrable and the integrable submanifolds are totally geodesic in M [8]. Moreover, from (3.14) we can see that the distributions D_0 and D_α are parallel [8]. So, using de Rham's decomposition theorem [6], we have M is a product of M_α and M_0, M_α corresponding to the integral manifold D_α and M_0 to that of D_0.

Thus, it is sufficient to show that M_α is a sphere in $\tilde{M}$ and M_0 is a totally geodesic submanifold in $\tilde{M}$.

We represent M_α by

$$x^h = x^h(u^a),$$

where u^a are local coordinates on M_α.

Thus we have

$$y^\kappa = y^\kappa(x(u)),$$

from which

(3.15) $$B_b{}^\kappa = B_b{}^h B_h{}^\kappa,$$

where

$$B_b{}^\kappa = \partial_b y^\kappa, \quad B_b{}^h = \partial_b x^h, \quad (\partial_b = \partial/\partial u^b).$$

Differentiating (3.15) covariantly along M_α, we find

$$\nabla_c B_b{}^\kappa = B_c{}^j B_b{}^i \nabla_j B_i{}^\kappa$$

because of $\nabla_c B_b{}^i = 0$, from which, using (3.10)

(3.16) $$\nabla_c B_b{}^\kappa = h_{ji} B_c{}^j B_b{}^i C^\kappa.$$

On the other hand, from (1.7) we see that

(3.17) $$\lambda = -G_{\kappa\lambda} F_\mu{}^\lambda D^\mu C^\kappa.$$

Differentiating (3.17) covariantly along M_α and taking account of (1.3) and (3.11), we find

$$\nabla_b \lambda = G_{\kappa\lambda}(F_\mu{}^\lambda D^\mu) B_b{}^j B_h{}^\kappa h_j{}^h,$$

from which, using (1.7),

$$\nabla_b\lambda=G_{\kappa\lambda}(-v^iB_i^\lambda-\lambda C^\lambda)h_j{}^hB_h{}^\kappa B_b{}^j=-\alpha v_jB_b{}^j$$

by virtue of (1.2) and (2.8).

From this we find

$$\nabla_c\nabla_b\lambda=-\alpha B_b{}^iB_c{}^j(\nabla_jv_i),$$

or, using (2.5), (3.9) and $l_j=0$,

$$\nabla_c\nabla_b\lambda=-\alpha^2\lambda g_{cb}$$

because $B_b{}^j$ are eigenvectors of $h_j{}^h$ with eigenvalue α. Thus, M_α is isometric with sphere in $\tilde{M}$.

Similarly, for M_0 we can see that $\nabla_cB_b{}^\kappa=0$ because of (3.16), from which, M_0 is totally geodesic in $\tilde{M}$.

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