A Class of Bilateral Generating Functions for the Jacobi Polynomial

H. M. SRIVASTAVA

Abstract

Put

$$(*)\qquad G[x,y]=\sum_{p,q=0}^{p+q\leq n}[-n]_{p+q}c_{p,q}x^py^q,$$

where $[\lambda]_m$ is the Pochhammer symbol and the $c_{p,q}$ are arbitrary constants. Making use of the specialized forms of some of his earlier results (see [8] and [9] the author derives here bilateral generating functions of the type

$$(**)\qquad \sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}{}_2F_1\left[\begin{matrix}\rho-n, & \alpha; \\ & \lambda+\rho;\end{matrix}\; x\right]G[y,z]t^n,$$

where α,ρ and λ are arbitrary complex numbers. In particular, it is shown that when $G[y,z]$ is a double hypergeometric polynomial, the right-hand member of (**) belongs to a class of general triple hypergeometric functions introduced by the author [7]. An interesting special case of (**) when $\rho=-m$, m being a nonnegative integer, yields a class of bilateral generating functions for the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ in the form

$$(***)\qquad \sum_{n=0}^{\infty}\binom{m+n}{n}P_{m+n}^{(\alpha-n,\beta-n)}(x)\;G[y,z]\;\frac{t^n}{n!},$$

which provides a unification of several known results.

Further extensions of (**) and (***) with $G[y,z]$ replaced by an analogous multiple sum $H[y_1,\cdots,y_m]$ are also discussed.

1. Introduction. In a recent paper we proved the generating relation [9, p. 71]

$$\sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}{}_{A+1}F_B\left[\begin{matrix}\rho-n, & (a); \\ & (b);\end{matrix}\; x\right]{}_{C+1}F_D\left[\begin{matrix}\sigma-n, (c); \\ (d);\end{matrix}\; y\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{[\lambda]_n[(a)]_n[(c)]_n}{n!\,[(b)]_n[(d)]_n}F^{(2)}\left[\begin{matrix}(a)+n:\rho;\lambda+n; \\ (b)+n:\text{———};\end{matrix}\; x,\frac{xt}{t-1}\right]$$

$$\cdot F^{(2)}\left[\begin{matrix}(c)+n:\sigma;\lambda+n; \\ (d)+n:\text{———};\end{matrix}\; y,\frac{yt}{t-1}\right]\left[\frac{xyt}{(1-t)^2}\right]^n, \qquad (1.1)$$

where $\qquad 0\leqq A\leqq B,\; 0\leqq C\leqq D,$

$$[\lambda]_n=\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1),\; n\geqq 1,\; [\lambda]_0=1,$$

Received by the editors March 20, 1971.

Notices Amer. Math. Soc. 16 (1969), pp. 1082–1083, Abstract 69T-B214. Supported in part by the National Research Council of Canada under Grant A7353.

${}_AF_B[z]$ is the generalized hypergeometric function defined by

$$ {}_AF_B\left[\begin{matrix}(a);\\(b);\end{matrix}\, z\right]=\sum_{n=0}^{\infty}\frac{[(a)]_n}{[(b)]_n}\frac{z^n}{n!} $$

$$ =\sum_{n=0}^{\infty}\frac{[a_1]_n[a_2]_n\cdots[a_A]_n}{[b_1]_n[b_2]_n\cdots[b_B]_n}\frac{z^n}{n!}, $$

(a) is taken to abbreviate the sequence of A parameters

$$ a_1,\ a_2,\ \cdots,\ a_j,\ \cdots,\ a_A, $$

that is, there are A of the a parameters and $[(a)]_n$ has the interpretation

$$ \prod_{j=1}^{A}[a_j]_n, $$

and similary for $(b),(c),(d),\cdots$ parameters, and $F^{(2)}[x,y]$ denotes Kampé de Fériet's double hypergeometric function [1, p. 150] in the elegantly contracted notation of Burchnall and Chaundy [2, p. 112].

Further extensions of the formula (1.1) are discussed in our subsequent paper [10], while its special case when $\rho=\sigma=0$ is incorporated as the bilinear generating relation (3.2), p. 309 in our earlier paper [8] motivated by an attempt to derive extensions of the well-known Hille-Hardy formula for Laguerre polynomials (cf., e.g., [6], p. 212).

In the limiting case when $y\to 0$ the formula (1.1) gives us

$$ \sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,{}_{A+1}F_B\left[\begin{matrix}\rho-n,\ (a);\\(b);\end{matrix}\, x\right]t^n $$

$$ =(1-t)^{-\lambda}\,F^{(2)}\left[\begin{matrix}(a):\ \rho;\ \lambda;\\(b):\text{——};\end{matrix}\, x,\ \frac{xt}{t-1}\right], \tag{1.2} $$

where, as before, $0\leqq A\leqq B$. For $A=B=1$ the double hypergeometric function on the right-hand side of (1.2) reduces to the first Appell function defined by

$$ F_1[\alpha,\ \beta,\ \beta';\ \gamma;\ x,\ y]=\sum_{m,n=0}^{\infty}\frac{[\alpha]_{m+n}[\beta]_m[\beta']_n}{[\gamma]_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, $$

where, for convergence of the double series,

$$ |x|<1,\ |y|<1; $$

and we readily have the elegant generating function

$$ \sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,{}_2F_1\left[\begin{matrix}\rho-n,\ \alpha;\\\gamma;\end{matrix}\, x\right]t^n $$

$$ =(1-t)^{-\lambda}\,F_1\left[\alpha,\ \rho,\ \lambda;\ \gamma;\ x,\ \frac{xt}{t-1}\right]. \tag{1.3} $$

Now let $\gamma=\rho+\lambda$ and make use of the known reduction formula [3, p. 238]

$$ F_1[\alpha,\ \beta,\ \beta';\ \beta+\beta';\ x,\ y]=(1-y)^{-\alpha}\,{}_2F_1\left[\begin{matrix}\alpha,\ \beta;\\\beta+\beta';\end{matrix}\ \frac{x-y}{1-y}\right]. $$

An appeal to Euler's transformation [3, p. 64]

$$ {}_2F_1\left[\begin{matrix}\alpha,\ \beta;\\\gamma;\end{matrix}\, z\right]=(1-z)^{-\alpha}\,{}_2F_1\left[\begin{matrix}\alpha,\ \gamma-\beta;\\\gamma;\end{matrix}\ \frac{z}{z-1}\right] $$

will then yield the formula

$$\sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,{}_2F_1\left[\begin{matrix}\rho-n,\ \alpha;\\ \lambda+\rho;\end{matrix}\ x\right]t^n=(1-x)^{-\alpha}(1-t)^{-\lambda}\cdot{}_2F_1\left[\begin{matrix}\alpha,\ \lambda;\\ \lambda+\rho;\end{matrix}\ \frac{x}{(1-x)(t-1)}\right], \tag{1.4}$$

whose specialized or limiting forms are scattered throughout the literature.

The object of the present paper is to discuss the usefulness of our formula (1.4) in the discovery of certain classes of generating functions. We first prove the generating relation.

$$\sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,{}_2F_1\left[\begin{matrix}\rho-n,\ \alpha;\\ \lambda+\rho;\end{matrix}\ x\right]F^{(2)}\left[\begin{matrix}-n,\ (a):\ (b);\ (b');\\ (c):\ (d);\ (d');\end{matrix}\ y,z\right]t^n$$
$$=(1-x)^{-\alpha}(1-t)^{-\lambda}\,F^{(3)}\left[\begin{matrix}\lambda::\text{---};\ (a);\text{---}:\ \ \alpha;\ (b);\ (b');\\ \text{---}::\text{---};\ (c);\text{---}:\ \lambda+\rho;\ (d);\ (d');\end{matrix}\ \frac{x}{(1-x)(t-1)},\ \frac{yt}{t-1},\ \frac{zt}{t-1}\right], \tag{1.5}$$

where $F^{(3)}[x,y,z]$ denotes a general triple hypergeometric series introduced by us in the form [7, p. 428]

$$F^{(3)}\left[\begin{matrix}(a)::(b);\ \ (b');\ \ (b''):(c);\ (c');\ (c'');\\ (e)::(g);\ \ (g');\ \ (g''):(h);\ (h');\ (h'');\end{matrix}\ x,y,z\right]$$
$$=\sum_{m,n,p=0}^{\infty}\frac{[(a)]_{m+n+p}\,[(b)]_{m+n}\,[(b')]_{n+p}\,[(b'')]_{p+m}\,[(c)]_m\,[(c')]_n\,[(c'')]_p}{[(e)]_{m+n+p}\,[(g)]_{m+n}\,[(g')]_{n+p}\,[(g'')]_{p+m}\,[(h)]_m\,[(h')]_n\,[(h'')]_p}\cdot\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!}. \tag{1.6}$$

Next of interest will be our attempt to give further extensions of (1.5) and to show that by appropriately specializing the parameters of Gauss's ${}_2F_1$, (1.5) can at once be reduced to a class of bilateral generating functions for the Jacobi polynomial defined by [6, p. 255]

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^{k}. \tag{1.7}$$

2. Proof of the Generating Function (1.5). On expressing the Kampé de Fériet function as a double series, if we make use of the elementary result

$$[-n]_{p+q}=\frac{(-1)^{p+q}\,n!}{(n-p-q)!},\quad 0\leqq p+q\leqq n,$$

and our formula (1.4), we shall find that

$$\sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,{}_2F_1\left[\begin{matrix}\rho-n,\ \alpha;\\ \lambda+\rho;\end{matrix}\ x\right]F^{(2)}\left[\begin{matrix}-n,\ (a):\ (b);\ (b');\\ (c):\ (d);\ (d');\end{matrix}\ y,z\right]t^n$$
$$=\sum_{n=0}^{\infty}[\lambda]_n\,{}_2F_1\left[\begin{matrix}\rho-n,\ \alpha;\\ \lambda+\rho;\end{matrix}\ x\right]t^n\sum_{p,q=0}^{p+q\leq n}\frac{[(a)]_{p+q}\,[(b)]_p\,[(b')]_q}{[(c)]_{p+q}\,[(d)]_p\,[(d')]_q}$$

$$\cdot \frac{(-y)^p(-z)^q}{p!q!(n-p-q)!}$$

$$= \sum_{p,q=0}^{\infty} \frac{[\lambda]_{p+q}\,[(a)]_{p+q}\,[(b)]_p\,[(b')]_q}{[(c)]_{p+q}\,[(d)]_q\,[(d')]_q} \frac{(-yt)^p}{p!} \frac{(-zt)^q}{q!}$$

$$\cdot \sum_{n=0}^{\infty} \frac{[\lambda+p+q]_n}{n!}\, {}_2F_1 \left[\begin{matrix} (\rho-p-q)-n,\ \alpha; \\ (\lambda+p+q)+(\rho-p-q); \end{matrix}\ x \right] t^n$$

$$=(1-x)^{-\alpha}(1-t)^{-\lambda} \sum_{p,q=0}^{\infty} \frac{[\lambda]_{p+q}\,[(a)]_{p+q}\,[(b)]_p\,[(b')]_q}{[(c)]_{p+q}\,[(d)]_p\,[(d')]_q}$$

$$\cdot \frac{[yt/(t-1)]^p}{p!} \frac{[zt/(t-1)]^q}{q!}\, {}_2F_1 \left[\begin{matrix} \alpha,\ \lambda; \\ \lambda+\rho; \end{matrix}\ \frac{x}{(1-x)(t-1)} \right]$$

$$= \sum_{m,p,q=0}^{\infty} \frac{[\lambda]_{m+p+q}\,[(a)]_{p+q}\,[\alpha]_m\,[(b)]_p\,[(b')]_q}{[(c)]_{p+q}\,[\lambda+\rho]_m\,[(d)]_p\,[(d')]_q}$$

$$\cdot \frac{[x/(1-x)(t-1)]^m}{m!} \frac{[yt/(t-1)]^p}{p!} \frac{[zt/(t-1)]^q}{q!},$$

which, in view of the definition (1.6), leads us immediately to the generating function (1.5).

3. Further Extensions and Particular Cases. A closer examination of the method of proof of the generating function (1.5), detailed in the preceding section, would suggest the existence of a 'mild' extension of (1.5) in which the Kampé de Fériet function $F^{(2)}[y,z]$ is replaced by the finite double sum

$$G[y,z] = \sum_{p,q=0}^{p+q\le n} [-n]_{p+q}\, c_{p,q}\, y^p z^q, \tag{3.1}$$

where the $c_{p,q}$ are arbitrary constants. Indeed it would also seem quite obvious that (1.5) can be extended to hold for an analogously terminating hypergeometric series in three and more variables in place of $F^{(2)}[y,z]$.

In general, it is of coures possible to replace the sum in (3.1) by a finite multiple sum in the form

$$H[y_1,\cdots,y_m] = \sum_{p_1,\cdots,p_m=0}^{p_1+\cdots+p_m\le n} [-n]_{p_1+\cdots+p_m}\, c_{p_1,\cdots,p_m}\, y_1^{p_1}\cdots y_m^{p_m}, \tag{3.2}$$

where the $c_{p_1,\cdots,p_m}$ are arbitrary constants. If the coefficients $c_{p_1,\cdots,p_m}$ in (3.2) are so specialized that the multiple sum can be interpreted as that of a terminating hypergeometric series in the m variables $y_1,\cdots,y_m$, then the right-hand side of the generating function (1.5) will evidently involve a hypergeometric function of the $m+1$ variables

$$\frac{x}{(1-x)(t-1)},\ \frac{y_1 t}{t-1},\ \cdots,\ \frac{y_m t}{t-1}.$$

Next we recall that for the Jacobi polynomials, defined by (1.7), we have [6, p. 255(7)]

$$P_n^{(\alpha,\beta)}(x) = \binom{2n+\alpha+\beta}{n}\left(\frac{x-1}{2}\right)^n {}_2F_1 \left[\begin{matrix} -n,\ -\alpha-n; \\ -\alpha-\beta-2n; \end{matrix}\ \frac{2}{1-x} \right],$$

whence it follows at once that

$$\binom{m+n}{n} P_{m+n}^{(\alpha-n,\beta-n)}(x) = (-1)^m \binom{2m+\alpha+\beta}{m} \frac{(-\alpha-\beta-m)_n}{n!}$$

$$\cdot \left(\frac{1-x}{2}\right)^{m+n} {}_2F_1 \left[\begin{matrix} -m-n, & -\alpha-m; \\ & -\alpha-\beta-2m; \end{matrix} \frac{2}{1-x} \right], \qquad (3.3)$$

where $$m=0,\ 1,\ 2,\cdots.$$

In view of the relationship (3.3), the special case $\rho=-m$ of our generating functio (1.5) leads us to the elegant result

$$\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-n,\beta-n)}(x)\, F^{(2)} \left[\begin{matrix} -n, & (a):\ (b);\ (b'); \\ & (c):\ (d);\ (d'); \end{matrix} y, z \right] t^n$$

$$= \binom{2m+\alpha+\beta}{m} \left(\frac{x+1}{x-1}\right)^{\alpha} \left(\frac{x+1}{2}\right)^{m} \delta^{\alpha+\beta+m}$$

$$\cdot F^{(3)} \left[\begin{matrix} -\alpha-\beta-m::-;\ (a);-: & -\alpha-m;\ (b);(b'); \\ \text{———}::-;\ (c);-:-\alpha-\beta-2m;\ (d);(d'); \end{matrix} \right.$$

$$\left. \frac{2}{(x+1)\delta},\ \frac{(x-1)yt}{2\delta},\ \frac{(x-1)zt}{2\delta} \right], \qquad (3.4)$$

which furnishes us with a class of bilateral generating functions for Jacobi polynomials and where, for the sake of brevity,

$$\delta=1+\frac{1}{2}(x-1)t.$$

It may be worth observing that by a suitable choice of parameters the function $F^{(2)}[y,z]$ in (3.4) can easily be reduced to the terminating hypergeometric series of Appell functions F_1, F_2, and F_4 (see [3], p. 224). Incidentally, in these three special cases the hypergeometric functions occurring on the right-hand side of (3.4) would become one or the other of the fourteen hypergeometric functions of three variables belonging to Lauricella's set and defined by Lauricella himself (cf. [4], p. 114), and we immediately arrive at the bilateral generating functions proved recently in [5, pp. 105-106].

References

[1] P. Appell et J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, Gauthier-Villars, Paris, 1926.

[2] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions*-II, Quart. J. Math. Oxford Ser. **12** (1941), 112-128.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, Vol. I, McGraw-Hill, New York-Toronto-London, 1953.

[4] G. Lauricella, *Sulle funzioni ipergeometriche a più variabili*, Rend. Circ. Mat. Palermo. **7** (1893), 111-158.

[5] H. L. Manocha, *Bilateral generating functions involving Jacobi polynomials*, Proc. Cambridge Philos. Soc. **66** (1969), 105-107.

[6] E.D. Rainville, *Special functions*, Macmillan, New York, 1960.

[7] H.M. Srivastava, *Generalized Neumann expansions involving hypergeometric functions*, Proc. Cambridge Philos. Soc. **63** (1967), 425-429.

[8] H.M. Srivastava, *An extension of the Hille-Hardy formula*, Math. Comp. **23** (1969), 305-311; see Notices Amer. Math. Soc. **15** (1968), 634-635.

[9] H.M. Srivastava, *Infinite series of certain products involving Appell's double hypergeometric functions*, Glasnik Mat. Ser. III. **4** (24) (1969), 67-73.

[10] H.M. Srivastava, *A formal extension of certain generating functions*, Notices Amer. Math. Soc. **16** (1969), 674; Glasnik Mat. Ser. III. **5** (25) (1970), 229-239.

University of Victoria, Victoria, British Columbia, Canada