A Note on Hopf Algebras

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1. Indroduction. Let K be a commutative ring with the identity 1. We shall define that an algebra (A, φ_A, η_A) over K (Note: Our algebras differ from those in [3] and [5]) is a graded K-module A together with morphisms of graded K-modules φ_A: $A\otimes A \longrightarrow A$ and η_A: $K \longrightarrow A$ such that the diagrams

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{1_A\otimes\varphi_A} & A\otimes A \\ {\scriptstyle \varphi_A\otimes 1_A}\downarrow & & \downarrow{\scriptstyle \varphi_A} \\ A\otimes A & \xrightarrow{\varphi_A} & A \end{array} \quad \text{and} \quad \begin{array}{ccccc} K\otimes A & \xrightarrow{\cong} & A & \xleftarrow{\cong} & A\otimes K \\ {\scriptstyle \eta_A\otimes 1_A}\downarrow & & \| & & \downarrow{\scriptstyle 1_A\otimes\eta_A} \\ A\otimes A & \xrightarrow{\varphi_A} & A & \xleftarrow{\varphi_A} & A\otimes A \end{array}$$

are commutative, where 1_A is the identity map and $\otimes$ is $\otimes_K$. (Note: $A=\{A_i \mid i=0,1,2,\cdots$ and each A_i is a K-module.$\}$)

In the above case, if $A_0 \cong K$ as K-modules then the algebra A is said to be connected. For a connected algebra A there exists a morphism of K-modules ε_A: $A \longrightarrow K$ such that

$$\varepsilon_A(a)=a \text{ if } a\in A_0 \text{ (i.e., } a\in K)$$
$$\varepsilon_A(a)=0 \text{ if } a\notin A_0.$$

Therefore, a connected algebra is an augmented algebra (p. 180 of [5]). Sometimes, the morphism φ_A is called the multiplication of A, and η_A is called the unit of A.

With the above commutative ring K, we can define coalgebras over K which are the dual concept for the algebras.

A coalgebra $(A, \Delta_A, \varepsilon_A)$ over K is a graded K-module A together with morphisms of graded K-modules

$$\Delta_A: A \longrightarrow A\otimes A \quad \text{and} \quad \varepsilon_A: A \longrightarrow K$$

such that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A\otimes A \\ {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle \Delta_A\otimes I_A} \\ A\otimes A & \xrightarrow{I_A\otimes\Delta_A} & A\otimes A\otimes A \end{array} \qquad \begin{array}{ccccc} A\otimes A & \xleftarrow{\Delta_A} & A & \xrightarrow{\Delta_A} & A\otimes A \\ {\scriptstyle \varepsilon_A\otimes I_A}\downarrow & & \| & & \downarrow{\scriptstyle I_A\otimes\varepsilon_A} \\ K\otimes A & \longrightarrow & A & \longleftarrow & A\otimes K \end{array}$$

are commutative, where $\otimes=\otimes_K$ and 1_A is the identity map of A. In this case the morphism Δ_A: $A \longrightarrow A\otimes A$ is called the comultiplication of A and ε_A is called the counit of A.

If $K \cong A_0$ is K-modules the coalgebra A is said to be connected, where $A=\{A_i\}_{i=0,1,2,\cdots\cdots}$. In this situation there exists the morphism of K-modules η_A: $K \longrightarrow A$ such that $\eta_A(k)=k$, where $k\in A_0=K$ and η_A is called an augmentation of A.

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K-modules

$$\varphi_A:\ A\otimes A\longrightarrow A,\qquad \eta_A:\ K\longrightarrow A$$
$$\Delta_A:\ A\longrightarrow A\otimes A,\qquad \varepsilon_A:\ A\longrightarrow K$$

such that

(i) $(A,\ \varphi_A,\ \eta_A)$ is an algebra over K with augmentation ε_A.

(ii) $(A,\ \Delta_A,\ \varepsilon_A)$ is a coalgebra over K with augmentation η_A.

(iii) the diagram

$$\begin{array}{ccccc}
A\otimes A & \xrightarrow{\varphi_A} & A & \xrightarrow{\Delta_A} & A\otimes A \\
\downarrow{\scriptstyle \Delta_A\otimes\Delta_A} & & & & \uparrow{\scriptstyle \varphi_A\otimes\varphi_A} \\
A\otimes A\otimes A\otimes A & & \xrightarrow{I_A\otimes T\otimes I_A} & & A\otimes A\otimes A\otimes A
\end{array}$$

is commutative, where the morphism T of K-modules is defined by

$$T(a_1\otimes a_2)=(-1)^{pq}(a_2\otimes a_1)$$

for $a_1\in A_p$ and $a_2\in A_q$, and $\otimes=\otimes_K$.

PROPOSITION 1. *In a Hopf algebra A over K, the morphism Δ_A is a morphism of algebras over K and the morphism φ_A is a morphism of coalgebras over K.*

Proof. We shall only prove the case of Δ_A. Since A is an algebra, so is $A\otimes A$ with multiplication.

$$A\otimes A\otimes A\otimes A\xrightarrow{1_A\otimes T\otimes 1_A}A\otimes A\otimes A\otimes A\xrightarrow{\varphi_A\otimes\varphi_A}A\otimes A.$$

By the condition (iii) above the diagram

$$\begin{array}{ccc}
A\otimes A & \xrightarrow{\varphi_A} & A \\
\downarrow{\scriptstyle \Delta_A\otimes\Delta_A} & & \downarrow{\scriptstyle \Delta_A} \\
A\otimes A\otimes A\otimes & \xrightarrow{(\varphi_A\otimes\varphi_A)(I_A\otimes T\otimes I_A)} & A\otimes A
\end{array}$$

is commutative. Furtheremore, since η_A is an augmentation of the coalgebra $(A,\Delta_A,\varepsilon_A)$ the diagram

$$\begin{array}{ccc}
K & \xrightarrow{\eta_A} & A \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \Delta_A} \\
K\otimes K & \xrightarrow{\eta_A\otimes\eta_A} & A\otimes A
\end{array}$$

is commuatative. Therefore we proved our assertion. (For definitions of morphisms of algebras and coalgebras, see §§1—2 of [7]) The study on Hopf algebras have been done in the papers [1], [2], [6] and [7] by several authors.

The objective of this paper is to prove a theorem on Hopf algebras over a commutative ring with 1 (see Theorem 1).

2. Direct limits. Let D be a partially ordered set. If for each pair $i, j \in D$ there exists $k \in D$ such that $i < k$ and $j < k$ then the partially ordered set D is called a directed set, where $<$ is the relation in D.

A direct system of sets $\{A, f\}$ over a directed set D is a function which attaches to each $i \in D$ a set A_i, to each pair i, j such that $i < j$ in D, a map

$$f(j,i):\ A_i \longrightarrow A_j$$

such that, for each $i \in D$

$$f(i,i) = \text{the identity map of } A$$

and for $i < j < k$ in D,

$$f(k,j)f(j,i) = f(k,i).$$

Let $\{A, f\}$ be a direct system over the directed set D of all positive integers and zero, where each A_i is a K-module and each $f(j,i)$ is a morphism of K-modules (A is as in § 1). Let ΣA be the direct sum of the K-modules of $\{A, f\}$, then there exist two morphisms of K-modules $i_l:\ A_l \longrightarrow \Sigma A$ and $P_l:\ \Sigma A \longrightarrow A_l$ such that

$$P_k i_l = 1_{A_l} \quad \text{if } l = k$$
$$P_k i_l = 0 \quad \text{if } l \neq k$$

If $a \in A_j$, we shall agree to identify a with its image in ΣA under the map i_j. For each $j < k$ in D and each $g_j \in A_j$ the element

$$f(k,j)(g_j) - g_j$$

of ΣA is called a relation. Let Q be the subgroup of ΣA generated by all relations. The direct limit $\varinjlim A_i$ of $\{A, f\}$ is defined by the factor group

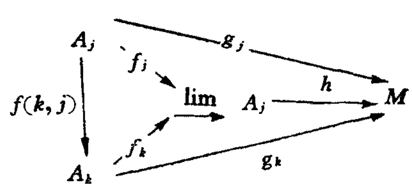
$$\varinjlim A_i = (\Sigma A)/Q,$$

where we have to know that ΣA and Q are K-modules and therefore $\varinjlim A_i$ is also a K-module. The natural morphism $\Sigma A \longrightarrow \varinjlim A_i$ of K-modules defines a morphism of K-modules

$$f_j:\ A_j \longrightarrow \varinjlim A_i,$$

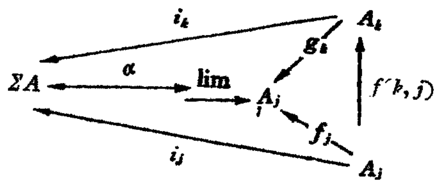
which are called projections. In this situation, if $j < k$ then $f_k \cdot f(k,j) = f_j$.

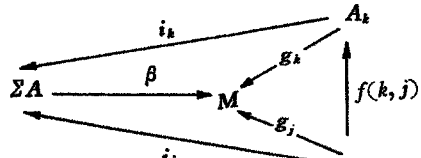
PROPOSITION 2. *Let $\varinjlim A_j$ be the direct limit of a direct system of K-modules. For each K-module M with morphisms $g_j : A_j \longrightarrow M$ of K-modules such that for each pair $j < k$ in D $g_k \cdot f(k,j) = g_j$,* there is a unique morphism $h:\ \varinjlim A_j \longrightarrow M$ of K-modules such that for $j \in D$ $g_j = hf_j$. (*i.e., each triangle in the following diagram is commutative.*



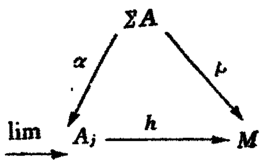
where $j < k$ in D.)

Proof. By the property of direct sums there exists a unique morphism $\alpha:\ \Sigma A \longrightarrow \varinjlim A_j$ and a unique morphism $\beta:\ \Sigma A \longrightarrow M$ of K-modules such that each triangle in the diagrams and





are commutative, where $j<k$ in D. Clearly, the kernel Q of α is contained in the kernel of β and therefore there is a morphism h: $\varinjlim A_j \longrightarrow M$ of K-modules such that the diagram



is commutative. The uniqueness of h is proved with the uniqueness of α and β.

3. Main Theorem. In this section, we shall assume that a fixed ring K is a commutative ring with identity 1. Let $A=\{A_i\}_{i=0,1,2,\dots}$ be a direct system of K-modules such that each A_i is a projective K-module. Then the direct sum ΣA of $\{A_i\}_{i=0,1,2,\dots}$ is a projective K-module (p. 6 of [3]), but the direct limit $\varinjlim A_i$ of $\{A_i\}$ is not necessarily a projective K-module.

PROPOSITION 3. *With the above conditions. If for each exact sequence*

$$M \xrightarrow{\alpha} N \longrightarrow 0$$

of K-modules and morphisms g_j: $A_j \longrightarrow N$, g_k: $A_k \longrightarrow N$ of K-modules such that $g_k \cdot f(k, j)=g_j$ there exist morphisms h_j: $A_j \longrightarrow M$, h_k: $A_k \longrightarrow M$ of K-modules with

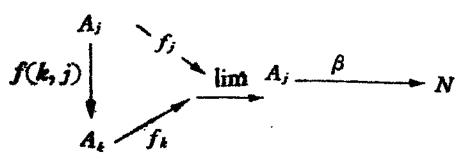
$$h_j=h_k f(k, j),\ g_j=\alpha h_j \text{ and } g_k=\alpha h_k,$$

then the direct limit $\varinjlim A_j$ is a projective K-module, where $j<k$.

Proof. Assume that there is a morphism β: $\varinjlim A_j \longrightarrow N$ of K-modules, where

$$M \xrightarrow{\alpha} N \longrightarrow 0$$

is an exact sequence of K-modules. Then, as in the following diagram



we can put

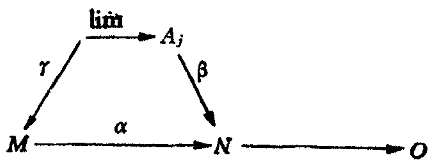
$$\beta f_j = g_j, \qquad \beta f_k = g_k.$$

In this case, $g_j = g_k f(k, j)$.

By our assumption there are two morphisms h_j: $A_j \longrightarrow M$, h_k: $A_k \longrightarrow M$ of K-modules such that

$$h_j = h_k f(k, j), \ g_j = \alpha h_j \text{ and } g_k = \alpha h_k,$$

where (j, k) is an arbitrary pair such that $j < k$. Therefore, by the proposition 2, there exists a morphism γ of K-modules such that the diagram



is commutative. This implies that $\varinjlim A_j$ is a projective K-module.

THEOREM 1. *If* $B = \{B_i\}_{i=0,1,2,\cdots}$ *is a connected Hopf algebra over the commutative ring* K *with* 1 *such that each* B_i, *for* $i = 0, 1, 2, \cdots$, *is a projective K-module satisfying the hypothesis of the proposition* 3, *then B is a direct limit of sub-Hopf algebras whose each underlying graded K-module is a direct summand of B.*

Proof. Suppose a sub-Hopf algebra $B' = \{B_i'\}_{i=0,1,2,\cdots}$ of the Hopf algebra B such that each B_i' is a summand of B_i for $i = 0, 1, 2, \cdots$. Then B' satisfies the conditions assumed on B. That is, $\{B', f|B'\}$ is a direct system and for an exact sequence

$$M \longrightarrow N \longrightarrow 0$$

of K-modules and g_j': $B_j' \longrightarrow N$, g_k': $B_k' \longrightarrow N$, if $g_k' \cdot f(k, j) | B_j' = g_j'$ then there are two morphisms h_j': $B_j' \longrightarrow M$ and h_k': $B_k' \longrightarrow M$ of K-modules such that

$$h_k' \cdot f(k, j) | B_j' = h_j',$$

where $j < k$ and $f(k, j)$ is a morphism in the direct system $\{B, f\}$.

Let $\mathscr{B}$ be the set of sub-Hopf algebras of B which are direct limits of sub-Hopf algebras whose underlying graded K-module is projective and a direct summand of B. Clearly $\mathscr{B}$ is closed under limits, thus there are maximal elements. Suppose that A is a maximal element of B. We want to prove that $A = B$. (Here, by the proposition 3, A is projective.)

Suppose that x is an element of the least degree of $B - A$. Now

$$\Delta_B(x) = x \otimes 1 + 1 \otimes x + y$$

where $y \in A \otimes A$. Without loss of generality we can assume that $y \in A^i \otimes A^i$ for $i = 0$ or positive integer, where $A = \varinjlim A^i$, and A^i is a sub-Hopf algebra whose underlying graded K-module is projective and a direct summand of B. Now if we assume that B^i is the subalgebra of B generated by A^i and x which is a direct summand of B, then B^i is a sub-Hopf algebra whose each K-module is projective. In this case $\varinjlim B^i$ contains x and A. This is impossible. Therefore $A = B$ and the proof is completed.

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