NOTES ON C-COMPACT SPACES AND FUNCTIONALLY COMPACT SPACES

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1. Introduction

It is well known that every continuous function on a compact space into a Hausdorff space is closed. In [1], G. Viglino showed that this property holds for a class of spaces (called C-compact spaces) which properly contains the class of compact spaces. R. F. Dickman Jr. and A. Zame [2] characterized the class of Hausdorff spaces with this property and called such spaces functionally compact spaces. These authors asked whether the C-compactness and the functional compactness are equivalent for Hausdorff spaces. In the present paper we characterize these two kinds of spaces and get partial answers of this question. (See Corollary 3 and Remark 2) The question (3) in [1] is answered negative. (See Remark 1)

An open filter base on a topological space X means a filter base consisting of open subsets of X.

DEFINITION. *A filter base on a topological space X converges to a subset A of X* if and only if every neighborhood of A contains a member of the filter base.

A topological space X is C-compact [1] if and only if given a closed subset Q of X and an open cover $\mathcal{O}$ of Q, then there exist a finite number of members of $\mathcal{O}$ whose closures cover Q.

A topological space X is functionally compact [2] if and only if whenever $\mathcal{U}$ is an open filter base on X such that the intersection A of the members of $\mathcal{U}$ is equal to the intersection of the closures of the members of $\mathcal{U}$, then $\mathcal{U}$ is a neighborhood base of A, or equivalently $\mathcal{U}$ converges to A. No separation axiom is assumed here.

For a subset A of a topological space X, A^- denotes the closure of A, A^0 denotes the interior of A, and A' denotes the complement $X \sim A$ of A.

2. Characterizations

THEOREM 1. *A topological space X is C-compact if and only if every open filter base $\mathcal{U}$ converges to $\bigcap\{U^- \mid U \in \mathcal{U}\}$.*

PROOF. Let $\mathscr{U}$ be an open filter base on a C-compact space X and let $A=\bigcap\{U^-|U\in\mathscr{U}\}$. If N is an open neighborhood of A, then

$$N'\subset A'=\bigcup\{U^{-\prime}|U\in\mathscr{U}\}.$$

By the C-compactness of X, $N'\subset\bigcup_{i=1}^{n}U_i^{-\prime-}$ for some finite $U_i\in\mathscr{U}$, $i=1,2,\cdots,n$. Thus

$$N\supset\bigcap_{i=1}^{n}U_i^{-\prime-\prime}\supset\bigcap_{i=1}^{n}U_i.$$

Hence $\mathscr{U}$ converges to A.

To prove the converse, let X be a topological space in which every open filter base converges to the set of cluster points of the open filter base. Suppose that there exist a closed subset Q of X and an open cover $\mathscr{O}$ of Q such that

$$Q\not\subset O_1^-\cup\cdots\cup O_n^-$$

for any finite number of members $O_i\in\mathscr{O}$, $i=1,2,\cdots,n$. Let $\mathscr{N}$ be the family of all open neighborhoods of Q and let $\mathscr{U}$ be the family of all $N\sim(\bigcup_{i=1}^{n}O_i)^-$ for every $N\in\mathscr{N}$ and for every finite number of members O_i of $\mathscr{O}$. Then we know that $\mathscr{U}$ is an open filter base on X. By the hypothesis $\mathscr{U}$ converges to A, where $A=\bigcap\{U^-|U\in\mathscr{U}\}$. On the other hand,

$$\begin{aligned}A&=\bigcap\{(N\sim(\bigcup_{i=1}^{n}O_i)^-)^-|N\in\mathscr{N},\ \textit{finite}\ O_i\in\mathscr{O}\}\\&\subset\bigcap\{N^-\cap(\bigcup_{i=1}^{n}O_i)^{-\prime-}|N\in\mathscr{N},\ \textit{finite}\ O_i\in\mathscr{O}\}\\&=\bigcap\{N^-|N\in\mathscr{N}\}\cap\bigcap\{(\bigcup_{i=1}^{n}O_i)^{-\prime-}|\textit{finite}\ O_i\in\mathscr{O}\}\\&\subset\bigcap\{N^-|N\in\mathscr{N}\}\cap\bigcap\{(\bigcup_{i=1}^{n}O_i)'|\textit{finite}\ O_i\in\mathscr{O}\}\\&\subset\bigcap\{N^-|N\in\mathscr{N}\}\cap Q'.\end{aligned}$$

Therefore Q' is an open neighborhood of A, and hence

$$Q'\supset N\sim(\bigcup_{i=1}^{n}O_i)^-$$

for some $N\in\mathscr{N}$ and some finite number of $O_i\in\mathscr{O}$, $i=1,2,\cdots,n$. But then

$$Q'\supset N\sim(\bigcup_{i=1}^{n}O_i)^-\supset Q\sim(\bigcup_{i=1}^{n}O_i)^-\neq\phi.$$

This is a contradiction.

A closed subset C of a topological space X is said to be r-closed [2] if whenever B is closed in C and $x\notin B$, there exist disjoint open sets in X containing x and B, respectively.

COROLLARY 2. *An r-closed subset of a C-compact space is C-compact.*

PROOF. It is proved similarly as Theorem 4 in [2].

COROLLARY 3. *For the spaces in which every closed subset A has the same neighborhood system as the intersection of all closed neighborhoods of A, the C-compactness is equivalent to the functional compactness.*

PROOF. It is clear from Theorem 1 that the C-compactness implies the functional compactness.

To prove the converse, let $\mathscr{U}$ be an open filter base and let $B=\cap\{U^-|U\in\mathscr{U}\}$. Let $\mathscr{N}$ be the family of all open neighborhoods of B and let $\mathscr{V}$ be the family of all $U\cup N$ for every $U\in\mathscr{U}$ and for every $N\in\mathscr{N}$. Then $\mathscr{V}$ is an open filter base and $\cap\mathscr{V}=\cap\{V^-|V\in\mathscr{V}\}=\cap\{N^-|N\in\mathscr{N}\}$. Putting $B^*=\cap\{N^-|N\in\mathscr{N}\}$, $\mathscr{V}$ converges to B^* by the functional compactness. Since B and B^* have the same open neighborhood base $\mathscr{N}$, $\mathscr{V}$ converges to B and hence $\mathscr{U}$ converges to B.

THEOREM 4. *A topological space X is functionally compact if and only if whenever given an open cover $\mathscr{O}$ of the complement A' of any closed subset A of X whose members' closures are disjoint from A, then for every neighborhood N of A there exist finite $O_i\in\mathscr{O}, i=1, 2, \cdots, n$ with $\bigcup_{i=1}^{n} O_i^-\supset N'$.*

PROOF. Let X be functionally compact and $\mathscr{O}$ be an open cover of the complement A' of a closed subset A of X such that $O^-\cap A=\phi$ for every $O\in\mathscr{O}$. Then

$$A\supset\cap\{O'|O\in\mathscr{O}\}\supset\cap\{O^{-\prime-}|O\in\mathscr{O}\}\supset\cap\{O^{-\prime}|O\in\mathscr{O}\}\supset A.$$

Hence

$$A=\cap\{O^{-\prime}|O\in\mathscr{O}\}=\cap\{O^{-\prime-}|O\in\mathscr{O}\}=\cap\{O'|O\in\mathscr{O}\}.$$

Case 1. $A=\phi$. $\mathscr{O}$ is an open cover of X. Since the functional compactness implies the generalized absolutely closedness, (A topological space is generalized absolutely closed [4] if and only if every open cover $\mathscr{O}$ of X has a finite subfamily whose union is dense in X, or equivalently every open filter base has a cluster point.) there exist finite $O_i\in\mathscr{O}, i=1, 2, \cdots, n$ such that $X=\bigcup_{i=1}^{n} O_i^-$.

Case 2. $A\neq\phi$. Let $\mathscr{U}$ be the family of all finite intersections of $O^{-\prime}$ for $O\in\mathscr{O}$. Then $\mathscr{U}$ is an open filter base on X and $A=\cap\mathscr{U}=\cap\{U^-|U\in\mathscr{U}\}$. By the functional compactness of X, $\mathscr{U}$ converges to A. Therefore for every neighborhood N of A, $N\supset\bigcap_{i=1}^{n} O_i^{-\prime}$ for some finite $O_i\in\mathscr{O}$. That is, $N'\subset\bigcup_{i=1}^{n} O_i^-$.

To prove the converse, let X be the topological space satisfying the condition of the theorem and let $\mathscr{U}$ be an open filter base on X with $\cap\mathscr{U}=\cap\{U^-|U\in\mathscr{U}\}(=A)$. Let an open neighborhood N of A be given. For each $x\in A'$, take a $U_x\in\mathscr{U}$ and a neighborhood N_x of x such that $N_x\cap U_x=\phi$, and hence $N_x^-\cap U_x=\phi$ and $N_x^-\cap A=\phi$.

Then there exist finite x_i, $i=1,2,\cdots,n$ such that $N'\subset\bigcup_{i=1}^{n}N_{x_i}^{-}$. Thus

$$N\supset\bigcap_{i=1}^{n}N_{x_i}^{-\prime}\supset\bigcap_{i=1}^{n}U_{x_i}.$$

Consequently $\mathscr{U}$ converges to A, and hence X is functionally compact.

A function $f:X\to Y$ is almost continuous [3] if and only if for each $x\in X$ and for each neighborhood V of $f(x)$ there exists a neighborhood U of x with $f[U]\subset V^{-0}$, or equivalently the inverse image of every regularly open subset of Y is open in X. ($V\subset Y$ is regularly open in Y if $V^{-0}=V$.)

A function $f:X\to Y$ is θ-continuous [5] if and only if for each $x\in X$ and for each neighborhood V of $f(x)$ there exists a neighborhood U of x with $f[U^{-}]\subset V^{-}$.

LEMMA 5. *Let f be a θ-continuous function on a generalized absolutely closed space X into a Hausdorff space Y. Then $f[X]$ is closed in Y.*

PROOF. Let $p\notin f[X]$. For each $x\in X$ choose an open neighborhood $U_{f(x)}$ of $f(x)$ with $p\notin U_{f(x)}^{-}$. By the θ-continuity of f there exists an open neighborhood N_x of x such that $f[N_x^{-}]\subset U_{f(x)}^{-}$. Since X is generalized absolutely closed,

$$X=\bigcup_{i=1}^{n}N_{x_i}^{-}$$

for some finite $x_i, i=1,2,\cdots,n$. Thus

$$f[X]=\bigcup_{i=1}^{n}f[N_{x_i}^{-}]\subset\bigcup_{i=1}^{n}U_{f(x_i)}^{-}.$$

Hence $Y\sim\bigcup_{i=1}^{n}U_{f(x_i)}^{-}$ is an open neighborhood of p and is disjoint from $f[X]$. Therefore $f[X]$ is closed.

LEMMA 6. *Every almost continuous function is θ-continuous.*

PROOF. Let $f:X\to Y$ be almost continuous. Let $x\in X$ and let U be an open neighborhood of $f(x)$. Then by the almost continuity of f there exists an open neighborhood V of x such that $f[V]\subset U^{-0}$. We show that $f[V^{-}]\subset U^{-}$ and complete the proof. If $f(x_0)\in U^{-\prime}$ for some $x_0\in V^{-}$, again by the almost continuity of f there exists an open neighborhood W of x_0 such that $f[W]\subset U^{-\prime-0}=U^{-\prime}$. [6, 1. E] But $W\cap V\neq\phi$, and hence $f[V]\cap U^{-\prime}\neq\phi$, which is contrary to the fact that $f[V]\subset U^{-0}\subset U^{-}$.

Lemma 6. is the answer of a question in Remark 3. 3 in [3].

As a generalization of Theorem 3 in [2], we have

THEOREM 7. *For the topological space X in which every two distinct point closures have disjoint neighborhoods, the following are equivalent.*

(i) *X is functionally compact.*

(ii) *Every almost continuous function on X into any Hausdorff space is closed.*

(iii) *Every continuous function on X into any Hausdorff space is closed.*

PROOF. (i)$\Rightarrow$(ii). Let f be an almost continuous function on a functionally compact space X into a Hausdorff space Y and let C be a closed subset of X. Suppose there exists a point y in $f[C]^{-}\sim f[C]$. Let $\mathscr{V}$ be the family of all open neighborhoods of y and let $\mathscr{U}=\{f^{-1}[V^{-0}]\mid V\in\mathscr{V}\}$. By Lemma 5 and Lemma 6 $f[X]$ is closed in Y, and hence $y\in f[X]$ and $f^{-1}[V^{-0}]\neq\phi$. Since f is almost continuous, $\mathscr{U}$ is an open filter base and $\bigcap\mathscr{U}=\bigcap\{U^{-}\mid U\in\mathscr{U}\}=f^{-1}[y]$. By the functional compactness of X, $\mathscr{U}$ is a neighborhood base of $f^{-1}[y]$. Since $X\sim C$ is an open subset of X containing $f^{-1}[y]$, there is a $U\in\mathscr{U}$ with $U\subset X\sim C$. But then $f[U]$ is an open neighborhood of y in $f[X]$ such that $f[U]\cap f[C]=\phi$, which is contrary to the fact that $y\in f[C]^{-}$. Thus $f[C]$ is closed in Y.

(ii)$\Rightarrow$(iii) is clear from the fact that every continuous function is almost continuous.

(iii)$\Rightarrow$(i). By the same method as in the proof of Theorem 3 in [2] we can construct a Hausdorff space Y and a continuous function on X onto Y which is not closed, if X is not functionally compact.

REMARK 1. The C-compactness is not productive even when every factor is a Hausdorff C-compact space.

Let X be the space of Example 2 in [1] and let $Y=[0,1]$ with the usual topology. Since Y is Hausdorff compact, Y is C-compact but $X\times Y$ is not C-compact. For the projection $\pi_2: X\times Y\rightarrow Y$ is continuous but not closed because $C=\left\{\left(\left(\frac{1}{n},0\right),\frac{1}{n}\right)\middle| n=1,2,\cdots\right\}$ is closed in $X\times Y$ but $\pi_2[C]=\left\{\frac{1}{n}\middle| n=1,2,\cdots\right\}$ is not closed in Y. This is the same method as in Remark in [2].

REMARK 2. Let $X=X_1\cup X_2$, where $X_1=\{x\mid 0\neq x\leq\omega^2,\ x=\text{ordinal number}\}$ and $X_2=\{\alpha_i\mid i=1,2,3,\cdots\}$, $\alpha_i\neq\alpha_j$ for $i\neq j$, $X_1\cap X_2=\phi$.

Topologize X as follows: X_1 has the order topology and each $\alpha_i\in X_2$ has $\{A_{ij}\mid j=1,2,3,\cdots\}$ as a neighborhood base, where $A_{ij}=\{\omega k+i\mid k>j,\ k=\text{natural number}\}$, $i=1,2,3,\cdots$.

Then X is T_1 functionally compact which is not C-compact.

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