# CERTAIN INTEGRALS OF LEGENDRE FUNCTIONS

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**Introduction**

Generalization of the classical Laplace transform

$$\phi(p)=p\int_0^\infty e^{-pt}f(t)dt,\ \text{Re } p>0,\ \text{or } \phi(p)\doteqdot f(t), \tag{1.1}$$

provided the integral on the right is convergent, has been given by Meijer [1] as

$$\phi(p)=p\int_0^\infty e^{-\frac{pt}{2}}\ w_{k+\frac{1}{2},m}(pt)(pt)^{-k-\frac{1}{2}}\ f(t)dt, \tag{1.2}$$

where $W_{k,m}$ is Whittaker's confluent hypergeometric function.

Let us denote (1.2) symbolically by

$$\phi(p)\ \xleftarrow[m]{k+\frac{1}{2}}\ f(t)$$

In each of (1.1) and (1.2), $f(x)$ is called the original of $\phi(p)$, and $\phi(p)$ the image of $f(x)$. If in (1.2), we put $k=m$, then on account of the identity

$$e^{-\frac{1}{2}px}\equiv(px)^{-\left(m+\frac{1}{2}\right)}W_{m+\frac{1}{2},m}(px),$$

(1.2) reduces to (1.1).

In this paper, certain integrals involving products of Legendre functions and also the operational representation of a few functions expressed by integrals have been given. The results given are believed to be new.

THEOREM 1.

*Let* (i) $\phi(p)\doteqdot f(x)$,

(ii) $\phi(p)\xleftarrow[\gamma+\frac{1}{2}]{\mu}x^{(\mu-1)}\ f(x)$.

*Then*

$$(a)\int_0^\infty \phi(t+a)\ (t+a)^{\left(\frac{\mu}{2}-1\right)}P_\gamma^\mu\left(1+\frac{2t}{a}\right)\frac{dt}{t^{\mu/2}}=a^{\mu-1}\ \phi(a), \tag{2,1}$$

$$(b)\ p^{\mu-1}\ \phi(p) \doteqdot \int_0^\infty f\left(\frac{x}{t+1}\right)(t+1)^{\frac{\mu}{2}-1} P_\gamma^\mu(1+2t)\frac{dt}{t^{\mu/2}}, \tag{2.2}$$

*Provided Re* $\mu<1$, $0>Re\ \gamma>-1$, *and* $f(t)$ *is bounded and absolutely integrable in* $(0,\infty)$.

PROOF. Let $\phi(p)\doteqdot f(x)$. Then

$$p\frac{\phi(p+a)}{(p+a)} \doteqdot e^{-ax}\ f(x) \tag{2.3}$$

Also we have

$$\left(1+\frac{a}{t}\right)^{\frac{\mu}{2}} p_\gamma^\mu\left(1+\frac{2t}{a}\right) \doteqdot e^{\frac{ap}{2}} w_{\mu,\gamma+\frac{1}{2}}(ap),\ \text{Re}\mu<1. \tag{2.4}$$

On using the relation(2.3) and (2.4) in Goldstein's theorem [2] : If $h_1(p)\doteqdot g_1(x)$ and $h_2(p)\doteqdot g_2(x)$, then $\int_0^\infty h_1(x)\ g_2(x)\frac{dx}{x}=\int_0^\infty g_1(x)\ h_2(x)\frac{dx}{x}$, we obtain (2.1).

Let $t=pt$, $a=p$, and $p>0$. We have from (2.1)

$$\int_0^\infty \phi[p(t+1)]\ (t+1)^{\frac{\mu}{2}-1}\ p_\gamma^\mu(1+2t)\ \frac{dt}{t^{\frac{\mu}{2}}}=p^{\mu-1}\ \phi(p). \tag{2.5}$$

Also $\phi(kp)\doteqdot f\left(\frac{x}{k}\right)$.

Then we get (2.2) from(2.5).

APPLICATIONS. Example 1. Let $f(x)=x^n K_m(bx)$, $\text{Re}(n\pm m)>-1$. Then we get from (2.1)

$$\int_0^\infty \frac{(t+a)^{\frac{\mu}{2}}}{t^{\frac{\mu}{2}}[(t+a)^2-b^2]^{\frac{n+1}{2}}} Q_n^m\left[\frac{(t+a)}{\{(t+a)^2-b^2\}^{1/2}}\right] P_\gamma^\mu\left(1+\frac{2t}{a}\right)dt$$

$$=\frac{[\sin\ (m+n)\ \pi]\,2^{(n+\mu-2)}\ a^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\ (\sin\ n\pi)\ \Gamma(n-m+1)\ b^{\left(n+\frac{1}{2}\right)}}$$
$$\times G_{44}^{42}\left[\frac{a^2}{b^2}\left|\begin{matrix}\frac{3}{4}-\frac{n}{2}\pm\frac{m}{2},\ \frac{1}{4}-\frac{\mu}{2},\ \frac{3}{4}-\frac{\mu}{2}\\ \frac{\gamma}{2}+\frac{3}{4},\ \frac{1}{4}-\frac{\gamma}{2},\ \frac{1}{4}+\frac{\gamma}{2},\ -\frac{\gamma}{2}-\frac{1}{4}\end{matrix}\right.\right], \tag{3.1}$$

$\operatorname{Re}(n\pm m)>-1$, $\operatorname{Re}\mu<1$, $\operatorname{Re}(n\pm m+\gamma+1,\ n\pm m-\gamma)>0$, $\operatorname{Re}\ m<0$.

We get from (2.2)

$$x^n\int_0^\infty (t+1)^{\left(\frac{\mu}{2}-n-1\right)} P_\gamma^\mu(1+2t)\, K_m\left(\frac{bx}{t+1}\right)\frac{dt}{t^{\frac{\mu}{2}}}$$

$$\doteqdot \frac{2^{(n+\mu-2)}\, p^{-\frac{1}{2}}}{\pi^{\frac{1}{2}}\, b^{n+\frac{1}{2}}}\, G_{44}^{42}\left[\frac{p^2}{b^2}\,\middle|\,\begin{matrix}\frac{3}{4}-\frac{n}{2}\pm\frac{m}{2}, & \frac{1}{4}-\frac{\mu}{2}, & \frac{3}{4}-\frac{\mu}{2} \\ \frac{\gamma}{2}+\frac{3}{4}, & \frac{1}{4}+\frac{\gamma}{2}, & \frac{1}{4}-\frac{\gamma}{2}, & -\frac{\gamma}{2}-\frac{1}{4}\end{matrix}\right], \tag{3.2}$$

$\operatorname{Re}\mu<1$, $\operatorname{Re}(n\pm m)>-1$, $\operatorname{Re}(n\pm m+\gamma+1,\ n\pm m-\gamma)>0$.

Taking appropriate $f(x)$ and using(2.1), we have the following important integrals. Operational representation of integrals similar to that of (3.2) can be obtained in each case.

**1.** $$\int_0^\infty (t+a)^{\frac{\mu}{2}-\frac{1}{2}}\left[p^n_{-\frac{1}{4}}\left\{\frac{\{(t+a)^2+b^2\}^{1/2}}{t+a}\right\}\right]^2 P_\gamma^\mu\left(1+\frac{2t}{a}\right)\frac{dt}{t^{\frac{\mu}{2}}}$$

$$=\frac{2^{3n+\frac{1}{2}}\,\Gamma\left[-2n\pm\left(\gamma+\frac{1}{2}\right)\right]\, a^{2n+\frac{1}{2}}}{b^{2n}[\Gamma(1-n)]^2\,\Gamma\left(\frac{1}{2}-2n-\mu\right)\,\Gamma\left(\frac{1}{2}-2n\right)}$$

$$\times {}_5F_4\left[\begin{matrix}\frac{1}{2}-n, & -n+\frac{\gamma}{2}+\frac{1}{4}, & -n-\frac{\gamma}{2}-\frac{1}{4}, & -n+\frac{\gamma}{2}+\frac{3}{4}, & -n-\frac{\gamma}{2}+\frac{1}{4}; \\ 1-n, & 1-2n, & \frac{1}{4}-\frac{\mu}{2}-n, & \frac{3}{4}-\frac{\mu}{2}-n; & \end{matrix}\ -\frac{b^2}{a^2}\right],$$

$\operatorname{Re}\ n<\frac{1}{4}$, $\operatorname{Re}\mu<1$, and $\operatorname{Re}\left\{-2n\pm\left(\gamma+\frac{1}{2}\right)\right\}>0$.

**2.** $$\int_0^\infty \frac{(t+a)^{\frac{\mu-1}{2}}}{[(t+a)^2+b^2]^{1/2}}\, P^n_{1/4}\left[\frac{\{(t+a)^2+b^2\}^{1/2}}{(t+a)}\right] P^n_{-1/4}\left[\frac{\{(t+a)^2+b^2\}^{1/2}}{(t+a)}\right]$$

$$\times P_\gamma^\mu\left(1+\frac{2t}{a}\right)\frac{dt}{t^{\frac{\mu}{2}}}=\frac{2^{\frac{3}{2}+3n}\,\Gamma\left(-2n+\gamma+\frac{3}{2}\right)\Gamma\left(-2n-\gamma+\frac{1}{2}\right)}{b^{(2n+1)}\, a^{\left(-2n+\frac{1}{2}\right)}\,[\Gamma(1-n)]^2\,\Gamma\left(\frac{3}{2}-\mu-2n\right)\Gamma\left(\frac{3}{2}-2n\right)}$$

$$\times {}_5F_4\left[\begin{matrix}\frac{1}{2}-n, & -n+\frac{\gamma}{2}+\frac{3}{4}, & -n-\frac{\gamma}{2}+\frac{1}{4}, & -n+\frac{\gamma}{2}+\frac{5}{4}, & -n-\frac{\gamma}{2}+\frac{3}{4}; \\ 1-n, & -2n+1, & \frac{3}{4}-\frac{\mu}{2}-n, & \frac{5}{4}-\frac{\mu}{2}-n: & \end{matrix} -\frac{b^2}{a^2}\right],$$

$$\mathrm{Re}\,\mu<1,\quad \mathrm{Re}(-2n+\gamma)>-\frac{3}{2},\quad \mathrm{Re}(-2n-\gamma)>-\frac{1}{2}.$$

3. $$\int_0^\infty (t+a)^{-\frac{\mu}{2}} Q_n\left\{\frac{(t+a)^2+2b^2}{2b^2}\right\} P_\gamma^\mu\left(1+\frac{2t}{a}\right)\frac{dt}{t^{\frac{\mu}{2}}}$$

$$=\frac{b^{2n+2}\,\pi}{(2a)^{2n+1}}\,\frac{\Gamma(2n+\gamma+2)\,\Gamma(2n-\gamma+1)}{\left[\Gamma\left(n+\frac{3}{2}\right)\right]^2\Gamma(2n-\mu+2)}$$

$$\times {}_5F_4\left[\begin{matrix} n+1, & n\pm\frac{\gamma}{2}+1, & n+\frac{\gamma}{2}+\frac{3}{4}, & n-\frac{\gamma}{2}+\frac{1}{2}; \\ n+\frac{3}{2}, & 2n+2, & n-\frac{\mu}{2}+1, & n-\frac{\mu}{2}+\frac{3}{2}; \end{matrix} -\frac{4b^2}{a^2}\right],$$

$\mathrm{Re}\ n>-1$, $\mathrm{Re}(2n+\gamma+2,\ 2n-\gamma+1)>0$, $\mathrm{Re}\ \mu<1$.

4. $$\int_0^\infty \left(1+\frac{a}{t}\right)^{-\frac{\mu}{2}} \{(a+t)^2-b^2\}^{-\frac{m}{2}} P_n^m\left(\frac{a+t}{b}\right) P_\gamma^\mu\left(1+\frac{2t}{a}\right) dt$$

$$=\frac{b^{m+\frac{1}{2}}\,a^{-\frac{1}{2}}\,2^{\mu-m-2}}{\pi\Gamma(-m-n)\Gamma(1+n-m)}$$

$$\times G_{44}^{42}\left[\frac{a^2}{b^2}\,\middle|\,\begin{matrix}\frac{m-n}{2}+\frac{3}{4}, & \frac{m+n}{2}+\frac{5}{4}, & -\frac{\mu}{2}+\frac{1}{4}, & -\frac{\mu}{2}+\frac{3}{4} \\ \frac{3}{4}+\frac{\gamma}{2}, & \frac{1}{4}\pm\frac{\gamma}{2}, & -\frac{\gamma}{2}-\frac{1}{4} & \end{matrix}\right],$$

$\mathrm{Re}\ m-1<\mathrm{Re}\ n<-\mathrm{Re}\ m$, $\mathrm{Re}\ \mu<1$, $\mathrm{Re}(m+\gamma+n)<1$, $\mathrm{Re}(-m+\gamma+n)>-1$, $\mathrm{Re}(-m+\gamma-n)>0$, and $\mathrm{Re}\ (-m-\gamma+n)>0$.)

THEOREM 2. *Let* (i) $\phi(x)\risingdotseq\phi(b)$, (4.1)

(ii) $x^{\frac{\lambda-3}{2}}\phi(x^{\frac{1}{2}})\risingdotseq G(p)$. (4.2)

(iii) $F(z)$ *be the Hankel transform of order* $\gamma$ *of* $x^{\left(-\mu+\frac{\lambda}{2}\right)}\phi\left(\frac{1}{x^{1/2}}\right)$. (4.3)

*Then*

$$\Gamma(\mu+\lambda)t^{\lambda-1}\int_0^\infty z^{\frac{1}{2}}(a^2t^4+z^2)^{-\frac{\mu}{2}}P_{\mu-1}^{-\gamma}\{at^2(a^2t^4+z^2)^{-\frac{1}{2}}\}$$

$$\times F(z)\,dz \doteqdot \frac{p^{3-\lambda}}{a}G\left(\frac{a}{p^2}\right), \tag{4.4}$$

*provided* $\phi(x)$ *and* $x^{2\mu-\lambda-3}\phi(x)$ *are continuous and absolutely integrable in* $(0,\infty)$.

$$\mathrm{Re}\left(\frac{\lambda}{2}-\mu+\gamma\right)>-\frac{3}{2},\ \mathrm{Re}\left(\mu-\frac{\lambda}{2},\ \lambda\right)>0,\ \mathrm{Re}\,\gamma>-\frac{1}{2}.$$

These conditions can be subsequently relaxed. It is sufficient if the definition integral exists as an absolutely convergent integral.

PROOF. Let $\phi(t)\doteqdot\phi(p)$ and $f(at)\doteqdot g\left(\frac{p}{a}\right)$. Using Goldstein's theorem, and interpreting, we get

$$\int_0^\infty \phi(x)\,f\left(\frac{t}{x}\right)\,\frac{dx}{x}\doteqdot\int_0^\infty f\left(\frac{x}{p}\right)\phi(x)\frac{dx}{x}, \tag{4.5}$$

Let $f(t)=t^{\lambda-1}e^{-at^2}$. We get from (4.5)

$$\int_0^\infty \phi(x)\,t^{\lambda-1}e^{(-at^2/x^2)}\frac{dx}{x^\lambda}\doteqdot\int_0^\infty\frac{x^{\lambda-2}}{p^{\lambda-1}}e^{(-ax^2/p^2)}\phi(x)\,dx \tag{4.6}$$

R. H. S. of (4.6) $=\dfrac{p^{3-\lambda}}{2a}G\left(\dfrac{a}{p^2}\right)$, where $G(p)\doteqdot x^{\frac{\lambda}{2}-\frac{3}{2}}\phi(\sqrt{x})$.

We have the integral [5 – p. 45]

$$\Gamma(\mu+\gamma)\int_0^\infty z(a^2t^4+z^2)^{-\frac{\mu}{2}}P_{\mu-1}^{-\gamma}\left[\frac{at^2}{(a^2t^4+z^2)^{1/2}}\right]J_\gamma\left(\frac{z}{x^2}\right)dz$$

$$=x^{4-2\mu}e^{(-at^2/x^2)},\ \mathrm{Re}\,\gamma>-1 \text{ and } \mathrm{Re}\,\mu>\frac{1}{2}.$$

L. H. S. of (4.6) $=\Gamma(\mu+\gamma)\,t^{\lambda-1}\displaystyle\int_0^\infty\phi(x)\frac{dx}{x^{\lambda-2\mu+4}}$

$$\times\int_0^\infty z(a^2t^4+z^2)^{-\frac{\mu}{2}}P_{\mu-1}^{-\gamma}\left[\frac{at^2}{(a^2t^4+z^2)^{1/2}}\right]J_\gamma\left(\frac{z}{x^2}\right)dz$$

$$=\frac{1}{2}\Gamma(\mu+\gamma)t^{\lambda-1}\int_0^\infty z^{\frac{1}{2}}\,(a^2t^4+z^2)^{-\frac{\mu}{2}}\,P_{\mu-1}^{-\gamma}\left[\frac{at^2}{(a^2t^4+z^2)^{1/2}}\right]\,F(z)dz,$$

using (4.3).

Hence we get (4.4).

APPLICATION. Let $\mu=\frac{\lambda}{2}+\frac{7}{4}$ in the theorem and

$$\phi(t)=(2bt)^{-\frac{1}{2}}\quad e\left(-\frac{t^2}{8b}\right)$$

Then we get from (4.4)

$$t^{(\lambda-1)}\int_0^\infty z(a^2t^4+z^2)^{\left(-\frac{\lambda}{2}-\frac{7}{8}\right)}\,P^{-\lambda}_{\frac{\lambda}{2}+\frac{3}{4}}\left[\frac{at^2}{(a^2t^4+z^2)^{1/2}}\right]\,J_r\left(\frac{z}{4b}\right)^{\frac{1}{2}}$$

$$\times K_\gamma\left(\frac{z}{4b}\right)^{\frac{1}{2}}\,dz \doteqdot \frac{\pi\Gamma(\lambda)p^{\frac{1}{2}}a^{\frac{1}{8}-\frac{\lambda}{4}}}{\Gamma\left(\frac{\lambda}{2}+\gamma+\frac{7}{4}\right)\;2^\lambda(a-2bp^2)^{\left(\frac{\lambda}{4}-\frac{1}{8}\right)}}\;P^{\frac{1}{4}-\frac{\lambda}{2}}_{-1/4}\left(\frac{a}{bp^2}-1\right),$$

Re $\gamma>-1$, Re $\lambda>0$.

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## REFERENCES

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