INTEGRALS INVOLVING WRIGHT'S GENERALIZED HYPERGEOMERIC FUNCTION AND H-FUNCTION OF FOX

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1. Introduction.

In this paper two integrals involving Wright's hypergeometric function have been evaluated in 2. In 3 two key integrals involving the product of Wright's hypergeometric function and H-function of Fox have been obtained by expressing H-function as Barnes type integral, interchanging the order of integrations and using integrals of 2. The integrals are very interesting, as on specialising the parameters, they yield many known and unkown results for Meijer's G-function, MacRobert's E-function, Wright's hypergeometric, Bessel-Maitland, Legendre, Whittaker and other related functions.

The following formulae will be required.

We shall denote the Wright's hypergeometric function [(8), p. 287].

$$\sum_{r=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 r)\cdots\Gamma(a_p+\alpha_p r)}{\Gamma(b_1+\beta_1 r)\cdots\Gamma(b_q+\beta_q r)}\cdot\frac{z^r}{r!} \tag{1.1}$$

symbolically as

$${}_p\phi_q\left[\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}: z\right],$$

where $\alpha_j>0, (j=1,2,\cdots,p);\beta_j>0,\ (j=1,2,\cdots,q)$.

$${}_p\phi_q\left[\begin{matrix}((a_p,1))\\((b_q,1))\end{matrix}: z\right]=\frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)}\,{}_pF_q\left[\begin{matrix}a_1,\cdots a_p\\b_1,\cdots b_q\end{matrix}: z\right]=E(p;a_r:q;b_s:-\frac{1}{z}). \tag{1.2}$$

The H-function introduced by Fox [(3), p. 408] and its asymptotic expansions and analytic continuation studied by Braaksma [(2), p. 278] will be represented and denoted as follows:

$$H_{m,n}^{h,l}\left[z\left|\begin{matrix}((\nu_m,c_m))\\((\delta_n,d_n))\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{h}\Gamma(\delta_j-d_j\xi)\prod_{j=1}^{l}\Gamma(1-\nu_j+c_j\xi)}{\prod_{j=h+1}^{n}\Gamma(1-\delta_j+d_j\xi)\prod_{j=l+1}^{m}\Gamma(\nu_j-c_j\xi)}z^{\xi}d\xi \tag{1.3}$$

where z is not equal to zero and empty product is interpreted as unity; h,l,m,n are integers satisfying $1\le h\le n,\ 0\le l\le m$; $c_j>0\ (j=1,2,\cdots,m)$, $d_j>0\ (j=1,2,\cdots,n)$ and $\nu_j\ (j=1,2,\cdots,m)$, $\delta_j\ (j=1,2,\cdots,n)$ are complex numbers such that no pole of

$\Gamma(\delta_j - d_j \xi)$ $(j=1,2,\cdots,h)$ coincides with any pole of $\Gamma(1-\nu_i+c_i\xi)$ $(i=1,2,\cdots,l)$ i.e.,

$$c_i(\delta_j+\sigma)\neq\nu_i-\rho-1$$

where $(\sigma,\rho=0,1,\cdots;\ j=1,2,\cdots,h;\ i=1,2,\cdots,l)$.

Further the contour runs from $\infty-i\sigma_1$ to $\infty+i\sigma_1$, such that the points,

$$\xi=\frac{(\delta_j+\sigma)}{d_j},\quad (j=1,\cdots,h;\ \sigma=0,1,\cdots),$$

which are poles of $\Gamma(\delta_j-d_j\xi)$, lie to the right and the points

$$\xi=\frac{(\nu_i-\rho-1)}{c_i},\quad (i=1,2,\cdots,l;\ \rho=0,1,\cdots),$$

which are the poles of $\Gamma(1-\nu_i+c_i\xi)$, lie to the left of the contour L.

From the equation (6.5) of Braaksma [(2), p. 279] we have

$$H_{m,n}^{h,l}\left[z\,\middle|\,\begin{matrix}((\nu_m,c_m))\\((\delta_n,d_n))\end{matrix}\right]=0(|z|^k),\quad \text{for small } z,$$

where $\sum_1^n(d_j)-\sum_1^m(c_j)\geqslant 0$ and $k=\mathscr{R}\left(\frac{\delta_i}{d_i}\right)$ $(i=1,2,\cdots,h)$.

(1.4) $$H_{m,n}^{h,l}\left[z^{-1}\,\middle|\,\begin{matrix}((a_m,\alpha_m))\\((b_n,\beta_n))\end{matrix}\right]=H_{n,m}^{l,h}\left[z\,\middle|\,\begin{matrix}((1-b_n,\beta_n))\\((1-a_m,\alpha_m))\end{matrix}\right].$$

(1.5) $$\Gamma(a+n)=\Gamma(a)\cdot(a)_n$$

(1.6) $$(\alpha)_{kn}=k^{nk}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)_n\cdots\left(\frac{\alpha+k-1}{k}\right)_n$$

(1.7) $$\Gamma(kz)=(2\pi)^{\frac{1}{2}(1-k)}\,k^{kz-\frac{1}{2}}\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{k}\right).$$

In what follows, the symbol $((a_r,\theta_r))$ represents the set of parameters (a_1,θ_1), $(a_2,\theta_2),\cdots,(a_r,\theta_r)$; $\alpha_1,\alpha_2,\cdots,\alpha_p,\beta_1,\beta_2,\cdots,\beta_q$, $c_1,c_2,\cdots,c_m,d_1,d_2,\cdots,d_n$ denotes positive numbers, $\mu=\sum_1^q\beta_j-\sum_1^p\alpha_j$ and $\lambda=\sum_1^h d_j-\sum_{h+1}^n d_j+\sum_1^l c_j-\sum_{l+1}^m c_j$.

2. In this section we have evaluated two integrals involving Wright's hypergeometric function,

(2.1) $$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_p\phi_q\left[\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}:\ cx^k(t-x)^s\right]dx$$

$$=t^{\alpha+\beta-1}{}_{p+2}\phi_{q+1}\left[\begin{matrix}((a_p,\alpha_p)),(\alpha,k),(\beta,s)\\((b_q,\beta_q)),(\alpha+\beta,k+s)\end{matrix}:ct^{k+s}\right],$$

where $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>0$, $\mu+1\geqslant 0$, k and s are positive numbers not both zero.

(2.2)
$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_p\phi_q\left[{}^{((a_p,\alpha_p))}_{((b_q,\beta_q))}:cx^k\right]dx$$
$$=\Gamma(\beta)t^{\alpha+\beta-1}{}_{p+1}\phi_{q+1}\left[{}^{((a_p,\alpha_p)),\,(\alpha,k)}_{((b_q,\beta_q)),\,(\alpha+\beta,k)}:ct^k\right],$$
where $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>0$, $\mu+1\geqslant 0$.

PROOF. The integral (2.1) can be established by putting $x=tv$, expressing Wright's hypergeometric function in the form of series, integrating term by term by [(5), p. 18, (1)], the result follows from (1.1). Proceeding as in (2.1), the integral (2.2) can be established. It can also be obtained from (2.1) by putting $s=0$.

3. In this section, two key integrals involving product of Wright's hypergeometric function and H-function have been obtained using (2.1) and (2.2).

(3.1)
$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_p\phi_q\left[{}^{((a_p,\alpha_p))}_{((b_q,\beta_q))}:cx^k(t-x)^s\right]H^{h,l}_{m,n}\left[zx^\delta(t-x)^d\left|{}^{((\nu_m,c_m))}_{((\delta_n,d_n))}\right.\right]dx$$
$$=t^{\alpha+\beta-1}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_jr)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_jr)}\frac{(ct^{k+s})^r}{r!}H^{h,l+2}_{m+2,n+1}\left[zt^{\delta+d}\left|{}^{(1-\alpha-kr,\delta),\,(1-\beta-sr,d),\,((\nu_m,c_m))}_{((\delta_n,d_n)),\,(1-\alpha-\beta-kr-sr,\delta+d)}\right.\right],$$
where $\mathscr{R}\left(\alpha+\delta\frac{\delta_i}{d_i}\right)>0$, $\mathscr{R}\left(\beta+d\frac{\delta_i}{d_i}\right)>0$, $(i=1,2,\cdots,h)$, $\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\pi$, $\mu+1\geqslant 0$.

(3.2)
$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_p\phi_q\left[{}^{((a_p,\alpha_p))}_{((b_q,\beta_q))}:cx^k\right]H^{h,l}_{m,n}\left[z\left(\frac{x}{t-x}\right)^\delta\left|{}^{((\nu_m,c_m))}_{((\delta_n,d_n))}\right.\right]dx$$
$$=t^{\alpha+\beta-1}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_jr)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_jr)}\frac{(ct^k)^r}{r!\,\Gamma(\alpha+\beta+kr)}H^{h+1,l+1}_{m+1,n+1}\left[z\left|{}^{(1-\alpha-kr,\delta),\,((\nu_m,c_m))}_{(\beta,\delta),\,((\delta_n,d_n))}\right.\right],$$
provided $\mathscr{R}\left(\alpha+\delta\frac{\delta_i}{d_i}\right)>0$. $(i=1,2,\cdots,h)$; $\mathscr{R}\left(\beta+\delta\frac{\nu_i}{c_i}\right)>-\delta$, $(i=1,2,\cdots,l)$, $\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\pi$, $\mu+1\geqslant 0$.

PROOF. To prove (3.1), substituting H-function as Barne-s integral using (1.3), interchanging the order of integrations, the integral then becomes
$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{h}\Gamma(\delta_j-d_j\xi)\prod_{j=1}^{l}\Gamma(1-\nu_j+c_j\xi)}{\prod_{j=h+1}^{n}\Gamma(1-\delta_j+d_j\xi)\prod_{j=l+1}^{m}\Gamma(\nu_j-c_j\xi)}z^\xi\left(\int_0^t x^{\alpha+d\xi-1}(t-x)^{\beta+d\xi-1}\right.$$

$${}_p\psi_q\left[\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}:cx^k(t-x)^s\right]dx\Big)d\xi$$

evaluating the inner integral using (2.1), expressing Wright's hypergeometric function in the form of series, this becomes

$$t^{\alpha+\beta-1}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j r)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j r)}\frac{(ct^{k+s})^r}{r!}$$

$$\times\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{h}\Gamma(\delta_j-d_j\xi)\prod_{j=1}^{l}\Gamma(1-\nu_j+c_j\xi)\Gamma(\alpha+rk+\delta\xi)\Gamma(\beta+sr+d\xi)}{\prod_{j=h+1}^{n}\Gamma(1-\delta_j+d_j\xi)\prod_{j=l+1}^{m}\Gamma(\nu_j-c_j\xi)\Gamma(\alpha+\beta+kr+sr+\delta\xi+d\xi)}(zt^{\delta+d})^{\xi}d\xi,$$

the result now follows from (1.3).

Proceeding as in (3.1), using (2.2) instead of (2.1), the integral (3.2) can be obtained.

Some deductions:

(a) Putting $s=0$ in (3.1), we get

$$(3.3)\quad\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_p\psi_q\left[\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}:cx^k\right]H^{h,l}_{m,n}\left[zx^{\delta}(t-x)^d\middle|\begin{matrix}((\nu_m,c_m))\\((\delta_n,d_n))\end{matrix}\right]dx$$

$$=t^{\alpha+\beta-1}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j r)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j r)}\frac{(ct^k)^r}{r!}H^{h,l+2}_{m+2,n+1}\left[zt^{\delta+d}\middle|\begin{matrix}(1-\alpha-kr,\delta)(1-\beta,d),((\nu_m,c_m))\\((\delta_n,d_n)),(1-\alpha-\beta-kr,\delta+d)\end{matrix}\right],$$

provided $\mathscr{R}\left(\alpha+\delta\frac{\delta_i}{d_i}\right)>0$, $\mathscr{R}\left(\beta+d\frac{\delta_i}{d_i}\right)>0$ $(i=1,2,\cdots,h)$, $\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\pi$ $\mu+1\geqslant0$.

(b) Setting $s=0$, $d=0$ in (3.1), and using (1.3) we have

$$(3.4)\quad\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_p\psi_q\left[\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}:cx^k\right]H^{h,l}_{m,n}\left[zx^{\delta}\middle|\begin{matrix}((\nu_m,c_m))\\((\delta_n,d_n))\end{matrix}\right]dx$$

$$=\Gamma(\beta)t^{\alpha+\beta-1}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j r)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j r)}\frac{(ct^k)^r}{r!}H^{h,l+1}_{m+1,n+1}\left[zt^{\delta}\middle|\begin{matrix}(1-\alpha-kr,\delta),((\nu_m,c_m))\\((\delta_n,d_n)),(1-\alpha-\beta-kr,\delta)\end{matrix}\right],$$

where $\mathscr{R}\left(\alpha+\delta\frac{\delta_i}{d_i}\right)>0$ $(i=1,2,\cdots,h)$, $\mathscr{R}(\beta)>0$, $\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\pi$, $\mu+1\geqslant0$.

(c) In (3.1), taking $s=d=0$, replacing $h,,l,m,n$, $((\nu_m,c_m))$ and $((\delta_n,d_n))$ by l, h, n,m, $((1-\delta_n,d_n))$ and $((1-\nu_m,c_m))$ respectively and using (1.3) and (1.4), we get

$$(3.5)\quad \int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_p\phi_q\left[\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}:cx^k\right]H^{h,l}_{m,n}\left[z^{-1}x^{-\delta}\left|\begin{matrix}((\nu_m,c_m))\\((\delta_n,d_n))\end{matrix}\right.\right]dx$$

$$=\Gamma(\beta)t^{\alpha+\beta-1}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j r)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j r)}\frac{(ct^k)^r}{r!}H^{l,h+1}_{n+1,m+1}\left[zt^{\delta}\left|\begin{matrix}(1-\alpha-kr,\delta),((1-\delta_n,d_n))\\((1-\nu_m,c_m)),(1-\alpha-\beta-kr,\delta)\end{matrix}\right.\right],$$

where $\mathscr{R}\left(\alpha+\delta\frac{\nu_i}{c_i}\right)>-\frac{\delta}{c_i}(i=1,2,\cdots,l)$, $\mathscr{R}(\beta)>0$, $\lambda>0$, $|\arg z^{-1}|<\frac{1}{2}\lambda\pi$, $\mu+1\geqslant 0$.

Several other integrals can be obtained by taking different combinations of the arguments and using (1.4).

4. Particular cases. By using the following known properties of the H-function,

$$(4.1)\qquad H^{h,l}_{m,n}\left[z\left|\begin{matrix}((a_m,1))\\((b_n,1))\end{matrix}\right.\right]=G^{h,l}_{m,n}\left(z\left|\begin{matrix}a_1,a_2,\cdots,a_m\\b_1,b_2,\cdots,b_n\end{matrix}\right.\right),$$

$$(4.2)\qquad H^{1,p}_{p,q+1}\left[z\left|\begin{matrix}((1-a_p,\alpha_p))\\(0,1),((1-b_q,\beta_q))\end{matrix}\right.\right]={}_p\phi_q\left[\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}:-z\right],$$

$$(4.3)\qquad H^{1,0}_{0,2}\left[z\left|\begin{matrix}-\\(0,1)(-\nu,\mu)\end{matrix}\right.\right]=J^{\mu}_{\nu}(z),$$

where $J_\nu^\mu(z)$ is Bessel-Maitland function [(7), p. 257]; the integrals (3.1) to (3.5) yield as particular cases many known and unknown results. However we mention here a few interesting known results.

In (2.1) and (2.2), putting $\alpha_j=1$ $(j=1,\cdots,p)$, $\beta_j=1$ $(j=1,\cdots,q)$, expressing the right hand side as generalised hypergeometric series using (1.1), (1.7) and (1.5), we get [(5), p. 104, (4)] and [(5), p. 104. (5)] respectively.

In (2.1), putting $t=1$, $\alpha_j=1$, $(j=1,\cdots,p)$, $\beta_j=1$, $(j=1,\cdots,q)$ replacing c by $-\frac{1}{c}$, using (1,1), (1.7) and (1.2), we obtain ((4), p. 234, (2)).

Choosing the parameters suitably, using (1.2) and (4.1), the results (3.2), (3.3) and (3,4) reduce to known results (2.4), (2.5) and (2.3) of [(1)] respectively.

Reducing the Wright's hypergeometric function to Gauss hypergeometic function, after little simplification in (3.2), (3.3) and (3.4), these reduce to results (2.4), (2.5), and (2.3) of [(6)].

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REFERENCES

[1] Bajpai, S. D., *Some integrals involving Gauss's hypergeometric function and Meijer's G-function*, Proc. Camb. Phil. Soc. 63, 1967.

[2] Braaksma, B. L. J., *Asymptotic expansions and analytic continuation*, Compos. Math. Vol. 15, Fasc. 3, 278, 1963

[3] Fox, C., *The G and H-functions as symmetrical Fourier Kernels*, Trans. Amer. Math. Soc. 98, 395-429, 1961.

[4] MacRobert, T. M., *Integrals involving E-functions*, Math. Zeitschr, 69, 234-226, 1958.

[5] Rainville, E. D., *Special functions, 2nd edition*, Macmillan, New York, 1963.

[6] Sharma, O. P., *Some integrals involving Gauss's hypergeometric function and H-function*. Collectanea Mathematica, 1965.

[7] Wright, E. M., *The asymptotic expansion of the generalised Bessel function*, Proc. Lond. Math. Soc. 38, 257, 1935.

[8] Wright, E. M., *The asymptotic expansion of the generalised hypergeometric function*, Jour. Lond. Math. Soc. 10, 237, 1935.