NON-ARCHIMEDEAN UNIFORMITIES

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Non-Archimedean uniformities were introduced by A. F. Monna in connection with normed vector spaces over non-Archimedean valued fields. Since then, they have drawn little attention. Recently, however, interest in functional analysis over such fields is awakening and it becomes apparent that more knowledge of non-Archimedean uniformities is needed. The same is true for zerodimensional topological spaces. They have lived an obscure existence, being considered as more or less pathological objects. Stone spaces are virtually the only kind of zero-dimensional spaces that have been studied. (An exception must be made for Banaschewski's paper [2]). But in functional analysis over non-Archimedean valued fields essentially all topological spaces are zerodimensional.

Thus it seems appropriate to delve deeper into the theory of non-Archimedean uniformities and zerodimensional topologies. The main purpose of this paper is to study the completion of a non-Archimedean uniform space.

1. Non-Archimedean uniformities.

We assume that the reader is familiar with the principles of the theory of uniform spaces ([3], [7]).

A *partition* of a set X is a collection of disjoint non-empty subsets of X that covers X. For a partition $\mathscr{U}$ of X and $x \in X$ let $\mathscr{U}[x]$ be the element of $\mathscr{U}$ that contains x. Further, for any partition $\mathscr{U}$ put

$$\mathscr{U}^{(2)} = \bigcup_{x \in X} \mathscr{U}[x] \times \mathscr{U}[x]$$

If $\mathscr{U}$ is a partition of X and $\mathscr{T}$ is a topology in X such that every element of $\mathscr{U}$ is $\mathscr{T}$-open, then every element of $\mathscr{U}$ is $\mathscr{T}$-closed, hence $\mathscr{T}$-clopen (*clopen*= closed and open): we call $\mathscr{U}$ a *$\mathscr{T}$-clopen partition* of X. If U is a uniformity on X, a *U-uniform partition* of X is a partition which is also a U-uniform cover. A uniformity U is said to be *non-Archimedean*([8]) if every U-uniform cover of X has a U-uniform refinement that is a partition. A collection Φ of partitions of X *generates* U if the finite intersections of the sets $\mathscr{U}^{(2)}$ ($\mathscr{U} \in \Phi$) form a base for U. In this way every collection of partitions of X generates a non-Archimedean uniformity.

Let U be a non-Archimedean uniformity in X, generated by a system Φ of partitions of X. We denote by $\mathscr{T}(U)$ the topology in X induced by U.

The elements of $\bigcup\Phi$ form a subbase for $\mathscr{T}(U)$. Every $\mathscr{U}\in\Phi$ is $\mathscr{T}(U)$-clopen. U is a Hausdorff uniformity iff for all $x, y\in X, x\neq y$ there exists a $\mathscr{U}\in\Phi$ with $\mathscr{U}[x]\neq\mathscr{U}[y]$.

A topology $\mathscr{T}$ in X is zerodimensional if the $\mathscr{T}$-clopen sets form a base for $\mathscr{T}$. Thus, the topology induced by a non-Archimedean uniformity is always zerodimensional.

EXAMPLE 1.1. As is well known, a metric d on X determines a Hausdorff uniformity on X a base of which is formed by the sets

$$\{(x,y)\in X\times X : d(x,y)\leq n^{-1}\},\ (n=1,2,\cdots)$$

This uniformity is non-Archimedean if d satisfies the *strong triangle inequality*

$$d(x,z)\leq\max(d(x,y),d(y,z)),\ (x,y,z\in X)$$

i.e. if d is an *ultrametric*.

A *valuation* on a field K is a function $n:K\to[0,\infty)$ such that for all $x,y\in K$,

$$n(x)=0 \text{ iff } x=0,$$
$$n(x+y)\leq n(x)+n(y),$$
$$n(xy)=n(x)n(y),$$

Such a valuation induces a metric d on K:

$$d(x,y)=n(x-y),\ (x,y\in K).$$

One can prove the following ([1]). If $n(2)>1$, then K is (isomorphic to) a subfield of the complexes and there exists a $\tau\in(0,1]$ such that $n(x)=|x|^{\tau}$ for all $x\in K$. If, on the other hand, $n(2)\leq 1$, then for all $x,y\in K$

$$n(x+y)\leq\max(n(x),n(y))$$

so that the induced metric is an ultrametric. In the latter case we call the valuation *non-Archimedean*.

EXAMPLE 1.2. Let G be a locally compact zerodimensional group. For any open subgroup H of G let $\mathscr{U}_H$ be the partition of G by the left cosets of H. These partitions generate a non-Archimedean uniformity U on G. By a theorem of Pontryagin ([6], (7.5)) every neighborhood of the neutral element of G contains an open subgroup of G. It follows that our U is just the ordinary left uniformity of the group G ([3], [7]).

EXAMPLE 1.3. Let $\mathscr{T}$ be a zerodimensional topology in X. The collection of all $\mathscr{T}$-clopen partitions generates a non-Archimedean uniformity, the ∞-*uniformity*. It is the strongest non-Archimedean uniformity that is compatible with the given topology $\mathscr{T}$. (This ∞-uniformity is the uniformity considered by A.F. Monna [8]).

EXAMPLE 1.4. Let $\mathscr{T}$ be a zerodimensional topology in X. For any infinite cardinal number $\aleph$ the clopen partitions of X that consist of fewer than $\aleph$ elements generate a non-Archimedean uniformity $U_{\aleph}$. It is compatible with the topology $\mathscr{T}$. The weakest of these uniformities is $U_{\aleph_0}$, which is precompact. If $\aleph > \mathrm{card}(X)$. $U_{\aleph}$ is the ∞-uniformity.

THEOREM 1.5. *If* $\dim X=0$, *the uniformity* $U_{\aleph_1}$ *in* X *is just the e-uniformity* (See [10]).

PROOF. We adopt the terminology of [9] and [10]. Let $\dim X=0$. We have to prove that any countable normal cover $\mathscr{U}$ of X has a refinement that is a countable clopen partition of X. By the observations made by Shirota in the beginning of [10] there exists a countable normal cover $\mathscr{V}$ such that $\mathscr{V}^{\Delta\Delta}<\mathscr{U}$. Then $\mathscr{W}=\{\mathrm{int} V : V\in\mathscr{V}\}$ is a countable open cover, and $\mathscr{W}^{*}<\mathscr{W}^{\Delta\Delta}<\mathscr{V}^{\Delta\Delta}<\mathscr{U}$. Let. $W_1, W_2, \cdots$ the elements of $\mathscr{W}$. For each i we have clo $W_i\subset S(W_i, \mathscr{W})$, so that we can choose a clopen Y_i with clo $W_i\subset Y_i\subset S(W_i, \mathscr{W})$ (remember that $\dim X=0$). Finally, let $Z_i=Y_i-(Y_1\cup\cdots\cup Y_{i-1})$. Then the non-empty Z_i form a countable clopen partition of X that refines $\mathscr{U}$.

EXAMPLE 1.6. Let X be a set and let $(\langle X_i, U_i\rangle)_{i\in I}$ be a family of non-Archimedean uniform spaces. For each i let $f_i: X\to X_i$. Then the weakest uniformity on X that makes every f_i uniformly continuous is non-Archimedean.

Special cases. If $(\langle X_i, U_i\rangle)_{i\in I}$ is a family of non-Archimedean uniform spaces, the product uniformity $\Pi_i U_i$ on $\Pi_i X_i$ is non-Archimedean. If $\langle X, U\rangle$ is a non-Archimedean uniform space and if $Y\subset X$, the relativization of U to Y is non-Archimedean.

2. Completion.

The following lemma is easy to prove.

LEMMA 2.1. *Let* $\langle X, U\rangle$ *be a uniform space. Let* Λ *be an index set: for each* $\lambda\in\Lambda$ *let* i_λ *be a uniformly continuous mapping of* $\langle X, U\rangle$ *into a complete Hausdorff uniform space* $\langle Y_\lambda, U_\lambda\rangle$. *Assume that* U *is the weakest uniformity on* X *that makes all the* i_λ *uniformly continuous. In* $Y=\Pi_\lambda Y_\lambda$ *we take the product uniformity and the corresponding topology (which is the product of the topologies we have in the* Y_λ). *Define* $i: X\to Y$ *by*

$$i(x)_\lambda=i_\lambda(x), \quad (x\in X; \lambda\in\Lambda).$$

Let $\tilde{X}$=clo $i(X)$ and let $\tilde{U}$ be the uniformity induced in X by the uniformity of Y. Then $\langle\tilde{X},\tilde{U}\rangle$ is the Hausdorff completion of $\langle X,U\rangle$.

Without proving this lemma we shall apply it to our non-Archimedean uniformities.

Let U be a non-Archimedean uniformity on a set X generated by a collection Φ of partitions of X. We consider each $\mathscr{U}\in\Phi$ as a uniform space, its uniformity being generated by the partition consisting of all one-element subsets of $\mathscr{U}$. We endow $\Pi\Phi=\prod_{\mathscr{U}\in\Phi}\mathscr{U}$ with the product uniformity: the induced topology is just the product topology. There is a natural map $i:X\to\Pi\,\Phi$ defined by

$$i(x)_{\mathscr{U}}=\mathscr{U}[x],\quad (x\in X;\mathscr{U}\in\Phi)$$

Let $\tilde{X}$=clo $i(X)$ and let $\tilde{\mathscr{U}}$ be the relativized uniformity on X. Then we find from the above lemma:

THEOREM 2.2. *$\langle\tilde{X},\tilde{U}\rangle$ is the Hausdorff completion of $\langle X,U\rangle$. In particular, the Hausdorff completion of a non-Archimedean uniform space is again non-Archimedean.*

COROLLARY 2.3. *Let X be a zerodimensional Hausdorff space and $\aleph$ an infinite cardinal number. The uniformity $U_\aleph$ of X is complete iff X is homeomorphic to a closed subset of some product $\prod_{\lambda\in\Lambda} D_\lambda$, where each D_λ is a discrete topological space with card $D<\aleph$*

PROOF. The "only if" is guaranteed by the preceding lemma. (Let Φ be the collection of all clopen partitions of X that have cardinality $<\aleph$). Conversely, let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of discrete spaces, card $D_\lambda<\aleph$ for each λ, and let X be a closed subspace of $\Pi_\lambda D_\lambda$. For each $\mu\in\Lambda$ let U_μ be the uniformity $U_\aleph$ in D_μ (Which is also the ∞-uniformity) and let f_μ be the restriction to X of the natural surjection $\Pi_\lambda D_\lambda\to D_\mu$. If U is the weakest uniformity on X that makes all the f_μ uniformly continuous, then U is complete. But this U is weaker than the uniformity $U_\aleph$ of X, and both uniformities induce the same topology in X. Hence, the uniformity $U_\aleph$ of X is also complete.

COROLLARY 2.4. *Let X be a zerodimensional space whose topology is Hausdorff. The uniformity $U_{\aleph_0}$ of X is complete iff X is compact. The uniformity $U_{\aleph_1}$ of X is complete iff X is homeomorphic to a closed subspace of a power of the space of all integers. If dim $X=0$, $U_{\aleph_1}$ is complete iff X is a Q-space* ([4], [10]).

The ∞-uniformity of X is complete iff X is homeomorphic to a closed subset of a product of discrete spaces. (In the terminology of [5] : iff X is C-compact, where C is the class of all discrete spaces).

For a zerodimensional space X let ζX be the topological space associated with the Hausdorff completion of the uniform space $\langle X, U_{\aleph_0}\rangle$. This ζX is a compact zerodimensional Hausdorff space and there exists a natural continuous map ζ of X onto a dense subset of ζX. If X, Y are zerodimensional spaces, every continuous $f:X\to Y$ is a uniformly continuous map of $\langle X, U_{\aleph_0}\rangle$ into $\langle Y, U_{\aleph_0}\rangle$. Therefore:

COROLLARY 2.5. *Let X, Y be zerodimensional spaces and let $f:X\to Y$ be continuous. Then there exists a unique continuous $\zeta f:\zeta X\to\zeta Y$ such that the following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ \downarrow & & \downarrow \\ \zeta X & \xrightarrow{\;\zeta f\;} & \zeta Y \end{array}$$

If, in addition, Y is compact and Hausdorff, then $\zeta Y=Y$, and the diagram

$$\begin{array}{ccc} X & \searrow^{f} & \\ \downarrow & & Y \\ \zeta X & \nearrow_{f} & \end{array}$$

is commutative.

Apparently, ζX is a zerodimensional analog of the Stone-Čech compactification of X. We call ζX the *Banaschewski compactification* of X (see [2]). ζX is the Stone space of the boolean ring of all clopen subsets of X. ζX is homeomorphic to the Stone-Čech compactification iff $\dim X=0$.

3. Homomorphism spaces.

In this section we demonstrate a very different method of constructing the completion of $\langle X, U\rangle$.

Let X, Y be topological spaces, F a set of continuous functions $X\to Y$. A function $\omega:F\to Y$ is a *homomorphism* if for every positive integer n and $f_1, \cdots, f_n\in F$,

$$(\omega(f_1), \cdots, \omega(f_n))\in \operatorname{clo}\{(f_1(x), \cdots, f_n(x)) : x\in X\}$$

the latter set being considered as a subset of Y^n.

For instance, let $*$ be a continuous operation $Y\times Y\to Y$. For $f, g\in F$ define $f*g$: $X\to Y$ by the formula $(f*g)(x)=f(x)*g(x)$. If $\omega:F\to Y$ is a homomorphism and if $f, g\in F$ are such that $f*g\in F$, then $\omega(f*g)=\omega(f)*\omega(g)$. For the proof of this statement, take $n=3$, $f_1=f$, $f_2=g$, $f_3=f*g$, and observe that $\{(y,z,y*z): y,z\in Y\}$ is a closed subset of Y^3.

If $y\in Y$ we use the symbol y also to indicate the constant function $X\to Y$ whose value is y. If Y is Hausdorff and if F contains the constants, then $\omega(y)=y$ for every homomorphism $\omega:F\to Y$.

Each $a\in X$ defines a homomorphism $a^*:F\to Y$ by

$$a^*(f)=f(a)\quad (f\in F)$$

a^* is called the *evaluation* at a.

THEOREM 3.1. *Let $\langle X,U\rangle$ and $\langle Y,V\rangle$ be non-Archimedean uniform spaces: assume V to be Hausdorff. Let F be the set of all uniformly continuous functions $X\to Y$. The following conditions on $\omega:F\to Y$ are equivalent.*

(i) *ω is a homomorphism.*

(ii) *For every $f\in F$, $\omega(f)\in clo\ f(X)$. If $f,g,h\in F$ and if for every $x\in X$ either $f(x)=g(x)$ or $f(x)=h(x)$, then either $\omega(f)=\omega(g)$ or $\omega(f)=\omega(h)$.*

(iii) *ω is a limit of evaluations at points of X, i.e.*

$$\omega\in clo\{a^*:a\in X\}$$

in the product topology of Y^F.

PROOF. The implications (iii)$\Rightarrow$(i)$\Rightarrow$(ii) are easy. (For (1)$\Rightarrow$(ii), observe that $\{(y,y,z): y,z\in Y\}\cup\{(y,z,y):y,z\in Y\}$ is closed. It remains to prove (ii)$\Rightarrow$(iii). To do this, we need a lemma.

LEMMA 3.2. *Let $\langle X,U\rangle,\langle Y,V\rangle$, F be as in Theorem* (3.1) *and assume that Y contains more than one element. Let $\omega:F\to Y$ satisfy* (ii) *of Theorem* (3.1). *Let $\mathscr{W}$ be the boolean algebra $\{U:U\subset X;\ \{U,X-U\}$ is a U-uniform partition of $X\}$, and put $\mathscr{W}_\omega=\{U\in\mathscr{W}: if\ f,g\in F$ and $f=g$ on U, then $\omega(f)=\omega(g)\}$. Then $\mathscr{W}_\omega$ is ultrafilter in $\mathscr{W}$, and $\omega(f)\in clo\ f(U)$ for all $f\in F$ $U\in\mathscr{W}_\omega$.*

PROOF. To prove that $\mathscr{W}_\omega$ is an ultrafilter we have to show it has the following four properties.

(α) $\phi\in\mathscr{W}_\omega$.

(β) If $U,V\in\mathscr{W}_\omega$, then $U\cap V\in\mathscr{W}_\omega$.

(γ) If $U\in\mathscr{W}$ and U contains an element of $\mathscr{W}_\omega$, then $U\in\mathscr{W}_\omega$

(δ) If $U\in\mathscr{W}$, then either $U\in\mathscr{W}_\omega$ or $X-U\in\mathscr{W}_\omega$.

Now (α) is easy (consider the constant functions) and (γ) is trivial. As to (β), take $f,g\in F$, $f=g$ on $U\cap V$. Define h by

$$h=f \text{ on } U,$$
$$=g \text{ on } X-U.$$

Then $h\in F$ and $\omega(f)=\omega(h)=\omega(g)$. Hence, $U\cap V\in \mathscr{W}_\omega$; we have proved ($\gamma$). Finally, take $U\in \mathscr{W}$, $U\not\in \mathscr{W}_\omega$. There exist $f_1, f_2\in F$ such that $f_1=f_2$ on U but $\omega(f_1)\neq\omega(f_2)$. Take $g_1, g_2\in F$, $g_1=g_2$ on $X-U$. Define $h\in F$ by

$$h=f_1=f_2 \text{ on } U,$$
$$=g_1=g_2 \text{ on } X-U.$$

Either $\omega(h)\neq\omega(f_1)$ or $\omega(h)\neq\omega(f_2)$: we may assume $\omega(h)\neq\omega(f_1)$. For $i=1,2$, we have for every $x\in X$ either $h(x)=f_1(x)$ or $h(x)=g_i(x)$: It follows that either $\omega(h)=\omega(f_1)$ or $\omega(h)=\omega(g_i)$. The first possibility being excluded we obtain $\omega(g_1)=\omega(h)=\omega(g_2)$. This proves ($\delta$); so $\mathscr{W}_\omega$ is an ultrafilter.

Take $f\in F$, $U\in \mathscr{W}_\omega$. We know $U\neq\phi$. Take $a\in U$. Define $g\in F$ by

$$g=f \quad \text{ on } U,$$
$$=f(a) \text{ on } X-U.$$

Then $\omega(f)=\omega(g)\in$ clo $g(X)=$clo $f(U)$.

Now we can prove the implication (ii)$\Rightarrow$(iii) of Theorem (3.1). Let ω satisfy (ii). Take $f_1,\cdots,f_n\in F$ and for each i let V_i be a neighborhood of $\omega(f_i)$. We have to prove

$$\bigcap_i f_i^{-1}(V_i)\neq\phi$$

There exists a finite V-uniform partition V of Y such that $\mathscr{V}[\omega(f_i)]\subset V_i$ for each i. For each i, $f_i^{-1}(\mathscr{V})=\{f_i^{-1}(V):V\in\mathscr{V}\}$ is a finite U-uniform partition of X: let $\mathscr{U}$ be a finite U-uniform partition of X that is a refinement of each $f_i^{-1}(V)$. From the lemma (obviously we may assume Y to contain more than only one element) it follows easily that $\mathscr{U}$ contains a $U\in\mathscr{W}_\omega$. For each i, $\omega(f_i)\in$ clo $f_i(U)$, so $f_i(U)\cap\mathscr{V}[\omega(f_i)]\neq\phi$. But by the definition of $\mathscr{V}$, $f_i(U)$ is contained in an element of $\mathscr{V}$. Therefore $U\subset f_i^{-1}(V_i)$. The observation $U\neq\phi$ completes the proof.

As a corollary we have the following theorem.

THEOREM 3.3. *Let X, Y be zerodimensional Hausdorff spaces; assume that Y contains more than one element. Let F be the set of all continuous maps $X\to Y$. The following conditions on X and Y are equivalent.*

(i) *Every homomorphism $F\to Y$ is an evaluation at a point of X.*

(ii) *X is homeomorphic to a closed subspace of some power of Y.* (See [5]).

PROOF. We endow X and Y with their ∞-uniformities (Ex. 1.3). Then F is just the space of all uniformly continuous maps $X \to Y$. The map $a \mapsto a^*$ is easily seen to be a homeomophism of X into Y^F. The implication (i)$\Rightarrow$(ii) follows directly from 1.1.

To prove (ii)$\Rightarrow$(i), let N be an index set and let X be a closed subspace of Y^N. For each $n \in N$ let $\bar{n}$ be the map $x \mapsto x(n)$ $(x \in X)$; then $\bar{n} \in F$. As $X \subset Y^N$ is closed, $\{(x, x^*);\ x \in X\}$ is a closed subset of $Y^N \times Y^F$. There is a map $\Phi: Y^F \to Y^N Y^F$ with

$$[\Phi(\omega)](n) = \omega(\bar{n}), \quad (n \in N)$$
$$[\Phi(\omega)](f) = \omega(f), \quad (f \in F).$$

As this Φ is continuous, $\{x^*:\ x \in X\} = \Phi^{-1}(\{(x, x^*):\ x \in X\})$ is a closed subset of Y^F. It follows from 3.1 that every homomorphism $F \to Y$ is an x^*.

Closely related to this corollary is the main theorem of this section.

THEOREM 3.4. *Let $\langle X, U \rangle, \langle Y, V \rangle$ be non-Archimedean uniform spaces: assume $\langle Y, V \rangle$ to be Hausdorff and complete. Let F be the set of all uniformly continuous functions $X \to Y$ and assume that U is the weakest uniformity on X that makes every element of F uniformly continuous. Then $\langle X, U \rangle$ is complete iff every homomorphism $F \to Y$ is an elaluation.*

Let X^* be the set of all homomorphisms $F \to Y$, provided with the uniformity U^* inherited from the product uniformity of Y^F. Then $\langle X^*, U^* \rangle$ is the Hausdorff completion of $\langle X, U \rangle$.

PROOF. For the second assertion of the theorem, apply 2.1, taking $\Lambda = F$, $Y_f = Y$, $V_f = V$, $i_f = f$ $(f \in F)$. The first part follows from the second.

The following is a useful generalization of the above theorems. We leave the proof to the reader.

THEOREM 3.5. *Let $\langle X, U \rangle$, $\langle Y, V \rangle$ be non-Archimedean uniform spaces. Let $\mathscr{B}$ be a covering of Y by closed sets such that the union of any two elements of $\mathscr{B}$ is contained in an element of $\mathscr{B}$. Let $F_{\mathscr{B}}$ be the set of all those uniformly continuous maps $X \to Y$ whose ranges are contained in elements of $\mathscr{B}$. The conclusions of* 3.1 *and* 3.4 *remain valid if we replace F by $F_{\mathscr{B}}$.*

One can prove a similar variant of 3.3. Instead of "X is homeomorphic to a closed subspace of some power of Y" one now obtains "X is homeomorphic to a closed subspace of a product of elements of $\mathscr{B}$", i.e. "$X \in K\mathscr{B}$" in the notation of [5].

COROLLARY 3.6. *Let X, Y be zerodimensional Hausdorff spaces: assume that*

Y contains more than one element. Let F be the collection of all continuous maps $X\to Y$ whose ranges have compact closure. Then the space of all homomorphisms $F\to Y$ (under the product topology) is the Banaschewski compactification of X.

By our definition a homomorphism $F\to Y$ has to commute with every continuous operation in Y. From the following examples we shall see that this condition may be relaxed considerably.

THEOREM 3.7. *Let X be a topological space. Let K be either a zerodimensional field or the (discrete) space Z of all integers or the (discrete) space N of all positive integers. Let F be the set of all continuous maps $X\to K$. Then a function $\omega: F\to K$ is a homomorphism iff $\omega(fg)=\omega(f)\omega(g)$, $\omega(f+g)=\omega(f)+\omega(g)$, $\omega(a)=a$ for all $f, g\in F$ and $a\in K$.*

PROOF. We only have to prove the "if". Assume $\omega(fg)=\omega(f)\omega(g)$, $\omega(f+g)=\omega(f)+\omega(g)$, $\omega(a)=a$ for all f, g, a: we prove that ω satisfies condition (ii) of (3.1). Let $f, g, h\in F$ be so that for all $x\in X$ either $f(x)=g(x)$ or $f(x)=h(x)$. Define $j: X\to X$ by

$$j=h \text{ on } \{x\in X\colon f(x)=g(x)\},$$
$$=g \text{ on } \{x\in X\colon f(x)=h(x)\}.$$

Then $j\in F$, $f+j=g+h$, $fj=gh$. Hence, $\omega(f)+\omega(j)=\omega(g)+\omega(h)$ and $\omega(f)\omega(j)=\omega(g)\omega(h)$. Therefore, $\omega(f)=\omega(g)$ or $\omega(f)=\omega(h)$.

Now take $f\in F$ and assume $\omega(f)\notin \operatorname{clo} f(X)$. Observe that $\omega(ag)=\omega(a)\omega(g)=a\omega(g)$ for all $g\in F$ and $a\in K$. In case K is a field, choose $g=[f-\omega(f)]^{-1}$. Then $g\in F$ and $fg=\omega(f)g+1$, hence $\omega(f)\omega(g)=\omega(f)\omega(g)+1$: contradiction. If $K=N$, put $g=[f-\omega(f)]^2$. Then $g\in F$ and $g=f^2-2\omega(f)f+\omega(f)^2$. Thus, $\omega(g)=\omega(f)^2-2\omega(f)^2+\omega(f)^2=0\notin K$: contradiction. Finally, if $K=Z$, put $g=[f-\omega(f)]^2-1$. Then $g\in F$, $\omega(g)=-1$, and $g\geq 0$. As is well known from elementary number theory, every non-negative integer is a sum of four squares of integers. It follows that there exist $g_1, g_2, g_3, g_4\in F$ such that $g=g_1^2+\cdots+g_4^2$. Then $-1=\omega(g)=\omega(g_1)^2+\cdots+\omega(g_4)^2$: contradiction.

NOTE. In case $K\subset(-\infty,\infty)$ or K is a field with a non-Archimedean valuation the same reasoning applies if we let F be the collection of all *bounded* continuous functions $X\to K$.

THEOREM 3.8. *Let X be a topological space, C a totally ordered set that is zerodimensional in its order topology, and F the set of all continuous maps (or all* *bounded continuous maps) $X \to C$. Then $w: F \to C$ is a homomorphism iff it is a lattice homomorphism and $w(a)=a$ for all $a \in C$.*

PROOF. Again we only prove the "if". Let $\omega: F \to C$ be a lattice homomorphism such that $\omega(a)=a (a \in C)$. The proof of the fact that ω satisfies the second half of condition (ii) in (3.1) is quite analogous to the first part of the proof of the preceding theorem; so we are done if $\omega(f) \in \text{clo} f(X)$ for all $f \in F$. Assume $\omega(f) \notin \text{clo} f(X)$ for certain $f \in F$. There exist $a, b \in C$ such that $a < \omega(f) < b$ while for every $x \in X$ either $f(x) \leqslant a$ or $f(x) \geqslant b$. For every $x \in X$ either $f(x) = f(x) \wedge a = (f \wedge a)(x)$ or $f(x) = (f \vee b)(x)$. Hence, either $\omega(f) = \omega(f \wedge a) = \omega(f) \wedge a = a$ or $\omega(f) = \omega(f \vee b) = \omega(f) \vee b = b$: contradiction.

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