HOMOTOPY FUNCTORS DETERMINED BY SET-VALUED MAPS

By Choon Jai Rhee

1. Introduction.

It is the purpose of this paper to investigate homotopy theories which can be applied to the study of set-valued maps. Given a family $X(\alpha)$ of subsets of a space X there is a homotopy functor. This functor is shown to be equivalent to the usual homotopy functor in certain cases. It is a consequence of the uniqueness theorem of the usual homotopy theory that theories defined are uniquely defined.

2. The Category M.

Let $(X, A; X(\alpha))$ be defined if X is a compact Hausdorff space, $X(\alpha)$ a collection of non-empty closed subsets of X such that each single point of X is also a member of $X(\alpha)$, and $A \in X(\alpha)$. For each pair of such triplets $(X, A; X(\alpha))$ and $(Y, B; Y(\beta))$, a map $F: (X, A; X(\alpha)) \to (Y, B; Y(\beta))$ is defined; if (1) $F: X \to Y$ is an upper semicontinuous function such that for each $u \in X(\alpha)$, $F(u) = \bigcup \{F(x) \mid x \in u\} \in Y(\beta)$, (2) for each $x \in A$, $F(x) = B$, (3) $F(x) \cap B \neq \phi$ implies $F(x) \supset B$ for $x \in X$. If $F: (X, A; X(\alpha)) \to (Y, B; Y(\beta))$ and $G: (Y, B; Y(\beta)) \to (Z, C; Z(r))$ are maps, then the composition $G \circ F$ is defined by $G \circ F(x) = \bigcup \{G(y) \mid y \in F(x)\}$ for each $x \in X$. The identity map $i: (X, A; X(\alpha)) \to (X, A; X(\alpha))$ is defined by $i(x) = x$ if $x \in X - A$, and $i(x) = A$ if $x \in A$.

THEOREM 2.1. *The collection of triplets $(X, A; X(\alpha))$ and maps form a category, call it M.*

We introduce a notion of homotopy in M. Let I denote the closed unit interval. Two maps F_0, F_1: $(X, A; X(\alpha)) \to (Y, B; Y(\beta))$ are said to be homotopic in M if there is an upper semicontinuous function $H: X \times I \to Y$ such that (1) for each $x \in X$, $H(x, 0) = F_0(x)$ and $H(x, 1) = f_1(x)$, (2) the restriction $H \mid X \times t$: $(X, A; X(\alpha)) \to (Y, B; Y(\beta))$ is a map in M for each $t \in I$. This homotopy relation is an equivalence relation.

3. Homotopy Functors.

Let I^n be the product of n unit intervals. Every point $x \in I^n$ is represented by $x = (x_1, \cdots\cdots, x_n)$, $x_i \in I$, $i = 1, \cdots\cdots, n$, and the boundary of I^n is denoted by ∂I^n.

For each object $(X, A; X(\alpha))$ in M, let $\Gamma^n(X, A; X(\alpha))$ be the collection of upper semicontinuous functions f: $I^n \to X$ such that for each $x \in I^n$, $f(x) \in X(\alpha)$, and for each $x \in \partial I^n$, $f(x)=A$, and if $f(x) \cap A \neq \phi$ then $f(x) \supset A$. Two members f_0 and f_1 of $\Gamma^n(X, A; X(\alpha))$ are said to be homotopic in $\Gamma^n(X, A; X(\alpha))$ if there is an upper semicontinuous function $h: I^n \times I \to X$ such that for each $x \in I^n$, $h(x, 0) = f_0(x)$ and $h(x, 1) = f_1(x)$ and the restriction $h|I^n \times t: I^n \to X$ is a member of $\Gamma^n(X, A; X(\alpha))$ for each $0 \leq t \leq 1$. This homotopy relation is also an equivalence relation in the set $\Gamma^n(X, A; X(\alpha))$ and partitions the set into equivalence classes. We denote the set of equivalence classes by $M\pi_n(X, A; X(\alpha))$ by $[f]$. For each pair of elements $f, g \in \Gamma^n(X, A; X(\alpha))$ define $h=f+g$ by

$$h(x_1, \cdots, x_n) = \begin{cases} f(2x_1, \cdots, x_n), & 0 \leq x_1 \leq \frac{1}{2} \\ g(2x_1 - 1, \cdots, x_n), & \frac{1}{2} \leq x_1 \leq 1 \end{cases}$$

and define $[f]+[g]$ to be $[f+g]$.

THEOREM 3.1. (1) *The operation + on the set* $M\pi_n(X, A; X(\alpha))$ *is independent of the choice of representative of elements of* $M\pi_n(X, A; X(\alpha))$, (2) *for* $n > 0$ $M\pi_n(X, A; X(\alpha))$ *is a group: the identity of* $M\pi_n(X, A; X(\alpha))$ *is the class of constant maps; the inverse of* $[f]$ *is represented by the element* $f^{-1}(x_1, \cdots\cdots, x_n) = f(1-x_1, \cdots\cdots, x_n)$ *for* $(x_1, \cdots\cdots, x_n) \in I^n$.

The proof is similar to that of theorems on ordinary homotopy groups [2].

We call an upper semicontinuous function f: $I \to X$ an M-path in X if $f(x) \in X(\alpha)$ for each $x \in I$. Then $M\pi_0(X, A; X(\alpha))$ is the set of M-path components of X with respect to $X(\alpha)$. Let F: $(X, A; X(\alpha)) \to (Y, B; Y(\beta))$ be a map in M. Then F induces a function $M\pi_n(F)$ of $M\pi_n(X, A; X(\alpha))$ into $M\pi_n(Y, B; Y(\beta))$ which is defined by $M\pi_n(F)[f] = [F \circ f]$ for each element $[f] \in M\pi_n(X, A; X(\alpha))$. Furthermore, it is seen that $M\pi_n(F)$ sends the zero elements of $M\pi_n(X, A; X(\alpha))$ to that of $M\pi_n(Y, B; Y(\beta))$.

THEOREM. 3.2. *For each* $n > 0$, $M\pi_n$ *is a covariant functor of the category* M *into the category* G *of groups.*

PROOF. Suppose $F: (X, A; X(\alpha)) \to (Y, B; Y(\beta))$ is a map in M.

Since $F\circ(f+g)=F\circ f+F\circ g$ for each pair $f, g\in\Gamma^n(X, A; X(\alpha))$, $M\ \pi_n(F)[f+g]=M\pi_n(F)[f]+M\pi_n(F)[g]$. If H_t: $(X, A; X(\alpha))\to(Y, B; Y(\beta))$ is a homotopy in M, $0\leq t\leq 1$, then $H_t\circ f$ is a homotopy in $\Gamma^n(Y, B; Y(\beta))$. Thus $M\pi_n(H_0)=M\pi_n(H_1)$. If F and G are maps in M such that $F\circ G$ is defined, then clearly $M\pi_n(F\circ G)=M\pi_n(F)\circ M\pi_n(G)$. Let $i:(X, A; X(\alpha))\to(X, A; X(\alpha))$ be the identity map. For each $f\in\Gamma^n(X, A; X(\alpha))$, we define a homotopy h_t in $\Gamma^n(X, A; X(\alpha))$ by taking $h_t(x)=f(x)$ if $x\in I^n$ and $t\neq 1$, and $h_1(x)=i\circ f(x)$, $x\in I^n$. Thus the identity map induces the identity map of $M\pi_n(X, A; X(\alpha))$.

REMARK (1) If each member of $X(\alpha)$ is a single point set in X, then $M\pi_n(X, A; X(\alpha))$ is the ordinary homotopy group $\pi_n(X, A)$ because every single-valued upper semicontinuous function is a continuous function. (2) If $(X, A; X(\alpha))$ and $(X, A; X(\beta))$ are objects in M such that the set $X(\alpha)$ is contained in $X(\beta)$, then there is a natural homomorphism from $M\pi_n(X, A; X(\alpha))$ to $M\pi_n(X, A; X(\beta))$, since every element of $\Gamma^n(X, A; X(\alpha))$ is an element of $\Gamma^n(X, A; X(\beta))$. (3) Since the set $\Gamma^n(X, X; X(\alpha))$ contains a single element, $M\pi_n(X, X; X(\alpha))=0$ for each n.

4. Uniqueness of the homotopy functors $M\pi_n$.

We give the compact-open topology to the set $\Gamma^n(X, A; X(\alpha))$. If K is a compact subset of I^n and U an open set in X, define $M[K, U]=\{f\in\Gamma^n(X, A; X(\alpha)) \mid f(K)\subset U\}$. The set of all $M[K, U]$ such that K is compact in I^n and U is an open set in X will be used as subbasic for the topology for $\Gamma^n(X, A; X(\alpha))$. Let $X^1_\alpha=\Gamma^1(X, A; X(\alpha))$ and A' be the constant member of X^1_α defined by $A'(x)=A$ for each $x\in I^n$, let $F^n(X^1_\alpha, A')$ be the set of all single-valued continuous functions f: $(I^n, \partial I^n)\to(X^1_\alpha, A')$. Since I^0 consists of a single element, we have $X^1_\alpha=F^0(X'_\alpha, A')$.

THEOREM 4.1. *Let F: $(X, A; X(\alpha))\to(Y, B; Y(\beta))$ be a map in M. Then F induces a continuous function F: $(X'_\alpha, A')\to(Y'_\beta, B')$ which is defined by $\hat{F}(s)=F\circ s$ for each $s\in X'_\alpha$.*

PROOF. Let $s\in X'_\alpha$ and $M[K, U]$ be a subbasis open set such that $F\circ s\in M[K, U]$.

Since $s(K)$ is compact in X and $F: X \to Y$ is upper semicontinuous, there is an open set V containing $s(K)$ such that $F(x) \subset U$ for each $x \in V$. Then $M[K, V]$ is a subbasic open set in X'_α such that $s \in M[K, V]$ and $F \circ s' \in M[K, U]$ for each $s' \in M[K, V]$.

The map $\hat{F}$: $(X'_\alpha, A') \supset (Y'_\beta, B')$ is continuous, and hence induces a homomorphism $\pi_n(\hat{F})$ of the ordinary homotopy group $\pi_n(X'_\alpha, A')$ into $\pi_n(Y'_\beta, B')$.

The proof of the following theorem is similar to that of theorem in oridinary homotopy theory [2].

THEOREM 4.2. *For each* $n \geq 1$, *the spaces* $\Gamma^n(X, A; X(\alpha))$ *and* $F^{n-1}(X'_\alpha, A')$ *are homeomorphic and the group* $M\pi_n(X, A; X(\alpha))$ *and* $\pi_{n-1}(X'_\alpha, A')$ *are isomorphic.*

We let θ_n: $\Gamma^n(X, A; X(\alpha)) \to F^{n-1}(X'_\alpha, A')$ be the natural homeomorphism provided by Theorem 4.2 and λ_n: $M\pi_n(X, A; X(\alpha)) \to \pi_{n-1}(X'_\alpha, A')$ the isomorphism induced by θ_n.

THEOREM 4.3. *Let* F: $(X, A; X(\alpha)) \to (Y, B; Y(\beta))$ *be a map in* M. *Then the following diagram commutes for each* $n \geq 1$.

$$\begin{array}{ccc} M\pi_n(X, A; X(\alpha)) & \xrightarrow{M\pi_n(F)} & M\pi_n(Y, B; Y(\beta)) \\ \downarrow \lambda_n & & \downarrow \lambda_n \\ \pi_{n-1}(X'_\alpha, A') & \xrightarrow{\pi_{n-1}(F)} & \pi_{n-1}(Y'_\beta, B') \end{array}$$

PROOF. Let $[f] \in M\pi_n(X, A; X(\alpha))$ be represented by f. Then $\hat{F}\ \theta_n(f) = \theta_n\ (F \circ f)$ and if h_t is a homotopy in $\Gamma_n(X, A; X(\alpha))$, then so is $\theta_n \circ h_t$ in $F^{n-1}\ (X'_\alpha, A')$.

THEOREM 4.4. *Let* $\{M\pi'_n, \lambda'_n\}_{n=0}$ *be a double sequence such that* (1) $M\pi'_n$ *is a covariant functor from* M *to the category* G *of groups,* $n > 0$ *and* $M\pi'_0$ *sends each object* $(X, A; X(\alpha))$ *of* M *to the set of* M*-path components of* X *with respect to* $X(\alpha)$, *and* (2) *for each object* $(X, A; X(\alpha))$ *in* M λ'_n: $M\pi'_n(X, A; X(\alpha)) \to \pi_{n-1}(X'_\alpha, A')$ *is an isomorphism* $n > 0$ *such that if* F: $(X, A; X(\alpha)) \to (Y, B; Y(\beta))$ *is a map in* M *then* $\pi_{n-1}(\hat{F})\ \lambda'_n \cdot M\pi_n(F)$. *Then there is a sequence* $\{h_n\}_{n=0}$

such that (1) *for each object* $(X, A; X(\alpha))$ *in* M, h_0: $M\pi_0(X, A; X(\alpha))\to M\pi_0'(X, A; X(\alpha))$ *is identity and* h_n: $M\pi_n(X, A; X(\alpha))\to M\pi_n'(X, A; X(\alpha))$ *is a one-to-one and onto function such that* $\lambda_n'\cdot h_n=\lambda_n$ *and* (2) *if* F: $(X, A; X(\alpha))\to(Y, B; Y(\beta))$ *is a map in* M, *then* $h_n\cdot M\pi_n(F)=M\pi_n'(F)\cdot h_n$.

PROOF. (1) Since $M\pi_0(X, A; X(\alpha))$ and $M\pi_0'(X, A; X(\alpha))$ are sets, each consisting of the set of M-path component with respect to $X(\alpha)$, they are the same sets. If $n>0$, we let $h_n=(\lambda_n')^{-1}\lambda_n$. (2) Since $\lambda_n\circ M\pi_n(F)=\pi_{n-1}(\hat{F})\cdot\lambda_n$ and $\lambda_n' M\pi_n'(F)=\pi_{n-1}(\hat{F})\cdot\lambda_n'$, the conclusion is immediate.

5. The Space $S^n(\alpha)$.

Let $S^n(\alpha)$ be the set of cellular subsets of the n-dimensional unit sphere S^n and $p\in S^n$. In [3], it is shown that $M\pi_n(S^n, p; S^n(\alpha))$ is isomorphic to the ordinary m-dimensional homotopy group $\pi_n(S^n, p)$ of S^n.

Now we give a topology to the set $S^n(\alpha)$: If U and V are open sets in S^n, define $N(U, V)=\{A\in S^n(\alpha)\mid A\subset U$ and $A\cap V\neq\phi\}$. The set of all $N(U, V)$ such that U and V are open sets in S^n will be used as a subbasis for the topology for $S^n(\alpha)$.

THEOREM 5.1. *For each* $m>0$, *the space* $S^n(\alpha)$ *has the same* m*-dimensional ordinary homotopy group as* S^n.

In the proof of this theorem, we will use the following two results in [1].

LEMMA 1. *For each* $F\in\Gamma^m(S^n, p; S^n(\alpha))$, *there is an element* $f\in F^m(S^n, p)$ *and a homotopy* H *in* $\Gamma^m(S^n, p; S^n(\alpha))$ *such that for each* $x\in I^m$, $H(x, 0)=F(x)$ *and* $H(x, 1)=f(x)$.

LEMMA 2. *If* f_0, $f_1\in F^m(S^n, p)$ *and* H *is a homotopy in* $\Gamma^m(S^n, p; S^n(\alpha))$ *relating* f_0 *to* f_1, *then there is a homotopy in* $F^m(S^n, \mathrm{p})$ *relating* f_0 *to* f_1.

PROOF OF THEOREM 5.1. We first prove that the space $S^n(\alpha)$ is path-wise connected. Suppose A and B are elements of $S^n(\alpha)$. Since A is cellular in S^n, there is a point $q\in S^n-A$. Let f: $I\to S^n(\alpha)$ be defined by $f(t)=\{(-t\cdot q+(1-t)\cdot y)/\|-t\cdot q+(1-t)\cdot y\|\ \mid y\in A\}$. Then $f(t)\in S^n(\alpha)$ for each $t\in I$. Now let $t_0\in I$ and $\{t_n\}_{n=1}$ be a sequence of points of I which converges to t_0. Suppose $f(t_0)\in N(U, V)$. Then $f(t_0)\subset U$ and there is a point $y_0\in A$ such that $(-t_0\cdot q+(1-t_0)y_0)/\|$

$-t_0q+(1-t_0)y_0 \| \in V$. Then there is a positive integer n_0 such that $f(t_n)\subset U$ and $(-t_n\cdot q+\cdot(1-t)y_0)/\| -t_nq+(1-t_n)y \| \in V$ for each $n\geq n$. Thus f is continuous at $t_0\in I$. Similarly, for the cellular set B, we find a point $p\in S^n-B$ and a continuous function g: $I\to S^n(\alpha)$ joining B to p. Let h: $I\to S^n$ be a continuous function joining q to p. Piecing f, g, and h property together, we find a continuous function $l:I\to S^n(\alpha)$ which joins A to B.

Since $S^n(\alpha)$ is piecewise connected, it is sufficient to show that $\pi_m(S^n(\alpha), p)$ and $\pi_m(S^n, p)$ are isomorphic where p is just a point of S^n. Let us note the followings: $F^m(S^n, p)\subset F^m(S^n(\alpha), p)\subset \Gamma^m(S^n, p; S^n(\alpha))$, and every homotopy in $F^m(S^n, p)$ is a homotopy in $F^m(S^n(\alpha), p)$ and every homotopy in $F^m(S^n(\alpha), p)$ is a homotopy in $\Gamma^m(S^n, p; S^n(\alpha))$.

Let α: $F^m(S^n, p)\to F^m(S^n(\alpha), p)$ be the inclusion map. Then α induces a homomorphism α^*: $\pi_m(S^n, p)\to\pi_m(S^n(\alpha), p)$, where α^* is defined by $\alpha^*[f]=[\alpha(f)]$ for $[f]\in F^m(S^n, p)$ and $[\alpha(f)]\in\pi_m(S^n(\alpha), p)$. Suppose the element $[F]\in\pi_m(S^n(\alpha), p)$ is represented by F. Then F is also an element of $\Gamma^m(S^n, p; S^n(\alpha))$. Let $f\in F^m(S^n, p)$ be an element which is given by Lemma 1 for F. Define $H(x, t)=\{(t\cdot f(x)+(1-t)\cdot y)/\| t\cdot f(x)+(1-t)\cdot y\| /y\in F(x)\}$ for $x\in I^m$ and $0\leq t\leq 1$. It is seen that H is a homotopy in $F^m(S^n(\alpha), p)$. Thus α^* is onto. If $\alpha^*[f]=\alpha^*[g]$, for $[f]$, $[g]\in\pi_m(S^n, p)$, then $\alpha(f)$ and $\alpha(g)$ are homotopic in $F^m(S^n(\alpha), p)$. Applying Lemma 2 $\alpha(f)$ and $\alpha(g)$ are also homotopic in $F^m(S^n, p)$. Therefore α^* is one-to-one.

It should be remarked that Professor W.L. Strother has shown in [4] that $\pi_m(C, p)=0$, where C is the set of all continua of S^n with the topology for C defined as above.

Wayne State University

Bibliography

[1] T. R. Brahana, M. K. Fort, Jr., and W. G. Horstman, *Homotopy for cellular set-valued functions*, Proc. Amer. Math. Soc. 16, 1965.

[2] S. T. Hu. *Homotopy theory*, Academic Press, 1959.

[3] C. J. Rhee, *Homotopy groups for cellular set-valued functions*, Proc. Amer. Math. Soc. 19, 1968.

[4] W. L. Strother. *Multi-homotopy*, Duke Math. J. 22, 1955.