IDEMPOTENT GENERATED REES MATRIX SEMIGROUPS

By Jin Bai Kim

1. J. Howie [7] defined an idempotent generated semigroup S to be a semigroup of which every element is expressible as a finite product of idempotents in S. Howie proved that in [7] the subsemigroup $T_X \backslash S_X$ of the full transformation semigroup T_X is an idempotent generated semigroup, where S_X is the symmetric group on a finite set X. Erdos [3] and Kim [8] independently proved that the multiplicative semigroup of all singular matrices in the matrix ring $M_n(F)$ of all n by n matrices over a field F is an idempotent generated semigroup. Preston [11] proved that for any finite set X and any finite positive integer r, a Rees factor semigroup I_{r+1}/I_r is completely 0-simple and has a representation into a Rees matrix semigroup, where I_r denotes the ideal of the full transformation semigroup T_X consisting of all elements of rank less than or equal to r. We have a similar result of Preston [11] for multiplicative matrix semigroups (see [1], p. 86).

Therefore it is interesting to know that what is a necessary and sufficient condition for a Rees matrix semigroup to be an idempotent generated semigroup? The object of this paper is to give such a necessary and sufficient condition.

2. For general notions about semigroups we refer to [1] and we shall follow the notation and terminology of [1].

DEFINITION. A subset B of a semigroup S is said to be an *IG set* if every element of S is expressible as a finite product of idempotents in S.

Let $S=M^0(G; I, J; P)$ be a Rees matrix semigroup over a group with zero G^0 with a sandwich matrix P. We define $(g)_{IJ}=\{(g)_{ij}: i\in I \text{ and } j\in J\}$, where g is a fixed element of the group G. We need the following lemma to prove Theorem 1.

LEMMA 1. *Let g_0 be the identity of the group G and let $(g_0)_{IJ}$ be an IG set in $S=M^0$ $(G;\ I,\ J;\ P)$. Then*

(i) *If an entry of P is g in G, then $(g)_{IJ}$ is an IG set. $(g^k)_{IJ}$ is also an IG set, where k is a positive integer.*

(ii) *If g in G is expressible as a finite product of entries of P, then $(g)_{IJ}$ is an IG set.*

PROOF. (i) Let g be an arbitrary element of G and we assume that there is an entry p_{ba} of P such that $p_{ba}=g$. Since $(g_o)_{IJ}$ is an IG set, for $(g_o)_{ib}$, there exists a set $\{p_{v_1u_1}, p_{v_1u_2}, p_{v_2u_2}, \cdots, p_{v_{k-1}u_k}, p_{v_ku_k}\}$ of nonzero entries of P such that $g_o=p_{v_1u_1}{}^{-1}p_{v_1u_2}p_{v_2u_2}{}^{-1}\cdots p_{v_{k-1}u_k}p_{v_ku_k}{}^{-1}$, where $u_1=i$ and $v_k=b$, that is, $(g_o)_{ib}=(p_{v_1u_1}{}^{-1})_{u_1v_1}\circ(p_{v_2u_2}{}^{-1})_{u_2v_2}\circ\cdots\circ(p_{v_ku_k}{}^{-1})_{u_kv_k}$, a product of idempotents. Similarly, we have that $(g_o)_{aj}$ is expressible as a finite product of idempotents. Then we have that $0\neq(g_0)_{ib}\circ(g_0)_{aj}=(g_0p_{ba}g_0)_{ij}=(g)_{ij}$, which shows that $(g)_{IJ}$ is an IG set. The following shows that $(g^k)_{IJ}$ is an IG set. $(g)_{ib}\circ(g_0)_{ab}\circ(g_0)_{ab}\circ\cdots\circ(g_0)_{aj}=(g^k)_{ij}$, in which the number of product signs $\circ$ is k-1.

(ii) Suppose that $g=p_{i_1j_1}\cdot p_{i_2j_2}\cdots p_{i_nj_n}$, where $p_{i_tj_t}$ are nonzero entries of P. Then we have that $(g)_{uv}=(g_o)_{ui_1}\circ(g_0)_{j_1i_2}\circ\cdots\circ(g_0)_{j_nv}$ which proves the (ii).

THEOREM 1. *$S=M^0(G;I,J;P)$ is an IG semigroup if and only if* (i) *$(g_o)_{IJ}$ is an IG set and* (ii) *P contains entries which generate the group G, where g_o is the identity of G.*

PROOF. The sufficiency of the theorem follows from Lemma 1.
(Necessity)(i) is clear. Suppose that there is an element g in G which is not expressible as a finite product of nonzero entries of P. Since S is an IG semigroup, we have that $(g)_{st}$ is expressible as a finite product of idempotents, say $(g)_{st}=(p_{j_1i_1}{}^{-1})_{i_1j_1}\circ\cdots\cdots\circ(p_{j_mi_m}{}^{-1})_{i_mj_m}$, where $s=i_1$ and $t=j_m$. Then $g=p_{j_1i_1}{}^{-1}\cdot p_{j_1i_2}\cdot p_{j_2i_2}{}^{-1}\cdots p_{j_mi_m}{}^{-1}$. This contra diction proves the part (ii) of the necessity.

3. To apply this Theorem 1 to, for example, full transformation semigroups, we need some additional study on Rees matrix semigroups.

DEFINITION. (1) A $n\times m$ matrix $P=(p_{ij})$ is called a *T matrix* if there exist two sequences $P_L=\{p_{i_jj};j=1,2,\cdots,m\}$ and $P_R=\{p_{ji_j};j=1,2,\cdots,n\}$ of nonzero entries of P such that $P_L\cap P_R$ contains one element and for each p_{st} of the sequences there exists p_{sv} or p_{ut} in $P_L\cup P_R$ such that $v\neq t$ and $u\neq s$.

(2) P is called a *$T(g)$ matrix* if P is a T matrix and every element of $P_L\cup P_R$ is equal to g.

LEMMA. 2. *If P is a $T(g_0)$ matrix, then $(g_0)_{IJ}$ is an IG set in $S=M^0(G;I,J;P)$, where g_o is the identity of the group G.*

PROOF. Let $(g_o)_{ij}$ be an element of S. We define $d((g_o)_{ij}, p_{st})=|j-s|+|i-t|$, where p_{st} is a nonzero entry of P with $p_{st}=g_o$. Let $d((g_o)_{ij})=\inf(d((g_o)_{ij}, p_{st}):$ $p_{st}=g_o)$. We prove this by induction on $d((g_o)_{ij})$. If $d((g_o)_{ij})=0$, then there exists p_{ji} with $p_{ji}=g_o$, and hence $(g_o)_{ij}$ is an idempotent. Suppose that we have been proved that every $(g_o)_{ij}$ is expressible as a finite product of idempotents if $d((g_o)_{ij})<n$. Now we assume that $d((g_o)_{ij})=n$. Then there exists a nonzero entry $p_{st}=g_o$ with $d((g_o)_{ij}, p_{st})=n$. There are two cases. (i) $s\neq j$ and $i\neq t$ and (ii) $i=t$ or $s=j$. Case (i). Choosing $(g_o)_{tj}$ and $(g_o)_{is}$, we have $0\neq(g_o)_{is}\circ(g_o)_{tj}$ $=(g_o p_{st} g_o)_{ij}=(g_o)_{ij}$. Since $d((g_o)_{tj})<d((g_o)_{ij})=n>d((g_o)_{is})$, by induction hypothesis, each of $(g_o)_{tj}$ and $(g_o)_{is}$ is expressible as a finite product of idempotents, so is $(g_o)_{ij}$. This proves the case (i). Case (ii). Assume that $d((g_o)_{ij},$ $p_{ju})=n$ for a nonzero entry $p_{ju}=g_o$ of P. We may assume that $u<i$. Since $d((g_o)_{i-1j}, p_{ju})=n-1$, we have that $(g_o)_{i-1j}$ is expressible as a finite product of idempotents by induction hypothesis. Since P is a $T(g_o)$ matrix, the column $i-1$ of P contains an element, say $p_{si-1}=g_o$. Then, for $(g_o)_{is}$, we have that $d((g_o)_{is},$ $p_{si-1})=1$, by induction hypothesis, $(g_o)_{is}$ is expressible as a finite product of idempotents. Thus we have that $0\neq(g_o)_{is}\circ(g_o)_{i-1j}=(g_o p_{si-1} g_o)_{ij}=(g_o)_{ij}$, a finite product of idempotents. This proves the Lemma 2.

The converse statement of Lemma 2 is not true by the following:

EXAMPLE. Let $G=S_2=\{e, (12)\}$ be the symmetric group on two letters $\{1, 2\}$, where e is the identity of G. Let $I=\{1, 2, 3, 4\}$ and $J=\{1, 2\}$. Let

$$P=\begin{bmatrix} 0 & e & (12) & (12) \\ e & e & e & 0 \end{bmatrix}.$$

Then in $M^0(G;I,J;P)$, $(e)_{IJ}$ is an IG set, but P is not a $T(e)$ matrix. This will be clarified by the following lemma.

LEMMA 3. *Let $S=M^0(G;I,J;P)$ be an IG semigroup. Then*

(i) *P is a T matrix.*

(ii) *There exist two invertible matrices U and V such that UPV is a $T(g_o)$ matrix, where g_o is the identity of G.*

(iii) *S and $M^0(G;I, J:UPV)$ are isomorphic.*

(iv) *The entries of UPV generate G.*

We omit the proof of Lemma 3 and we refer to Corollary 3.12 of [1] for the proof of Lemma 3—(iii).

Now we have our main theorem of this note.

THEOREM 2. *$S=M^0(G;I,J;P)$ is an IG semigroup if and only if* (i) *there exist invertible matrices U and V such that $UPV=T(g_o)$ and* (ii) *the entries of UPV generate the group G.*

PROOF. The necessity of Theorem 2 follows from Lemma 3 and the sufficiency follows from Theorem 1.

It is not hard to prove the following theorem. (See [7].)

THEOREM 3. *Let I_r be the ideal of a full transformation semigroup T_X on a finite set X consisting of all elements of rank less than or equal to r. If $S=M^0(G;I,J;P)$ is a representation of a Rees factor semigroup I_{r+1}/I_r, then there exist invertible matrices U and V such that* (i) *$UPV=T(g_o)$ and* (ii) *entries of UPV can generate G, where g_o is the identity of G.*

It is also not hard to see that:

If $M^0(G;I,J;P)$ is an *IG* semigroup, then it is regular.

We raise the following questions:

QUESTION 1. What are necessary and sufficient conditions for a semigroup to be an *IG* semigroup?

QUESTION 2. Is an idempotent generated semigroup regular?

4. It might be worth to read a different proof of Theorem ([8], [3]) in comparison with a proof of Erdos, and I shall give my original proof of the follwing theorem.

THEOREM. *Let $M_n(F)$ be the set of all $n\times n$ matrices over a field F. Every singular matrix in $M_n(F)$ is a product of idempotent matrices in $M_n(F)$. Denoting the set of all singular matrices in $M_n(F)$ by $S_n(F)$, we have that $S_n(F)$ is an idempotent generated multiplicative semigroup.*

PROOF. (i) Let $A\in S_n(F)$. It is well knwon that any matrix A is similar to the Jordan canonical matrix $J=(J_1,J_2,\cdots,J_t)$ of A, where J_i $(i=1,2,\cdots,t)$ is the companion matrix of an invariant factor of the characteristic matrix $xI-A$ of A.

It is sufficient to prove that J is a product of idempotent matrices in $M_n(F)$. Since A is singular, there exists k in $(1,2,\cdots,t)$ such that J_k takes the form

$$J_k=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & a & b & c & \cdots & d \end{bmatrix}.$$

I_{n_i} denotes the identity of $M_{n_i}(F)$. Letting $J_i\in M_{n_i}(F)$ $(i=1,2,\cdots,t)$ we have that $J=D_2D_1$, where $D_1=\mathrm{diag}(J_1,J_2,\cdots,J_{k-1},0,I_{n_k-1},J_{k+1},\cdots,J_t)$ and $D_2=\mathrm{diag}(I_{n_1},I_{n_2},\cdots,I_{n_{k-1}},J_k,I_{n_{k+1}},I_{n_t})$.

(ii) We have that $J_k=LN$ and $N=A_{n_k}A_{n_k-1}\cdots A_2E$, where

$$L=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ a & b & c & \cdots & d & 0 \end{bmatrix},$$

$E=\mathrm{diag}(0,I_{n_k-1})$, and $A_i=\mathrm{diag}\left(I_{i-2},\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix},I_{n_k-i}\right)$ for $i=2,3,\cdots,n_k$.

we can check that L,E, and A_i $(i=2,3,\cdots,n_k)$ are iden potents and hence we can show that D_2 is a product of idempotent matrices.

(iii) Now consider D_1. We have that $D_1=B_1B_2\cdots B_{k-1}EB_{k+1}\cdots B_t$,

$$B_i=\begin{cases}\mathrm{diag}(I_{n_1},\cdots,I_{n_{i-1}},J_i,I_{n_{i+1}},\cdots,I_{n_{k-1}},E,I_{n_{k+1}},\cdots,I_{n_t}) & \text{if } i\in(1,2,\cdots,k-1),\\ \mathrm{diag}(I_{n_1},\cdots,I_{n_{k-1}},E,I_{n_{k+1}},\cdots,I_{n_{i-1}},J_i,I_{n_{i+1}},\cdots I_{n_t}) & \text{if } i\in(k+1,k+2,\cdots,t).\end{cases}$$

We shall show that $B_i(i\neq k)$ is a product of idempotents.

(iv) We have

$$E_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},$$

and

$$E_2=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}1&1&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

We notice that each E_i $(i=1,2)$ is a product of four idempotent matrices.

(v) We see that

$$G_1=\begin{pmatrix} x & y & z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x-1 & y & z & 0 \end{pmatrix},$$

$$G_2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & x-1 & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G_8=G_4G_3G_1=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \quad G_9=G_6G_5G_2=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & x & y & z \end{bmatrix}$$

where $G_3=\mathrm{diag}(G_7,1,0)$, $G_4=\mathrm{diag}(1,G_7,0)$, $G_5=\mathrm{diag}(0,G_7,1)$, $G_6=\mathrm{diag}(0,1,G_7)$,

and $G_7=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

Using the result of (iv), we see that each G_i $(i=3,4,5,6)$ is a product of idempotents and so is G_j $(j=1,2,8,9)$.

(vi) Applying the generalized results of (iv) and(v) to B_i in (iii), we see that B_i is a product of idempotents in $M_n(F)$ and so is D_1. This completes the proof.

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