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Some Remarks on Dimensions of Topological Spaces

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1. **Introduction.** This paper is a rather brief account to investigate the method to decide dimensions of topological spaces with some restriction. We give, first, three conditions, each of which is equivalent to the fact that a metric space has dimension $\leqq n$. If a metric space R is the sum of the closed subsets F_i, $i=0, 1, \cdots, k$, and f_i continuous mappings of F_i into S^n, we obtain a sufficient condition in order that a continuous extension of f_0 over R exist. Finally we also give a necessary and sufficient condition in order that a hereditarily paracompact space have strong inductive dimension $\leqq n$.

Terminology and notations are based on J. Nagata [2].

2. **Dimension of metric spaces.** Throughout this section all spaces are metric unless the contrary is explicitly stated.

LEMMA 1. *Let* $\mathfrak{B}=\{V_\alpha|\alpha<\tau\}$ *be an open collection of a topological space* R *such that* $\mathrm{ord}\, B(\mathfrak{B})\leqq n$. *Let* $F_0=\bar{V}_0$ and $F_\alpha=\bar{V}_\alpha-\cup\{V_\beta|\ \beta<\alpha\}$ *for* $\alpha<\tau$. *Then* $\mathcal{F}=\{F_\alpha|\alpha<\tau\}$ *is a closed collection with* $\mathrm{ord}\,\mathcal{F}\leqq n+1$.

Proof. The closedness of F_α is clear. Let p be a point of R. We suppose $p\in V_\alpha$ and $p\notin V_\beta$ for every $\beta<\alpha$. Then we have $p\notin F_\gamma$ for every $\gamma>\alpha$. If $p\in F_\beta$ for some $\beta<\alpha$, then it follows from $p\notin V_\beta$ that $p\in\bar{V}_\beta-V_\beta=B(V_\beta)$. Therefore we have $\mathrm{ord}\,\mathcal{F}\leqq n+1$ by virtue of $\mathrm{ord}\, B(\mathfrak{B})\leqq n$.

It is well known that covering dimension (dim) and strong inductive dimension (Ind) coincide for general metric spaces. We also note that R has dimension $\leqq n$ if and only if for any locally finite open covering $\mathcal{U}$ of R there exists a locally finite open covering $\mathfrak{B}$ with $\mathrm{ord}\,\mathfrak{B}\leqq n+1$, $\mathfrak{B}<\mathcal{U}$ (Theorem II.6 of [2]). This theorem was originally proved for any normal space by C. H. Dowker [1].

Now we are ready to show conditions to decide dimension of metric spaces and this will be of some interest in the relation between dimension and the locally finite open collections.

THEOREM 1. *In a metric space R the following conditions are equivalent.*

(1) $\dim R=\mathrm{Ind}\, R\leqq n$.

(2) *For every locally finite open collection* $\{U_\gamma|\gamma<\tau\}$ *and closed collection* $\{F_\gamma|\gamma<\tau\}$ *in* R *satisfying* $F_\gamma\subset U_\gamma$ *for every* $\gamma<\tau$, *there exists an open collection* $\{V_\alpha|\alpha<\tau\}$ *such that*

$$F_\gamma\subset V_\gamma\subset\bar{V}_\gamma\subset U_\gamma,$$
$$\mathrm{ord}\{B(V_\gamma)\,|\,\gamma<\tau\}\leqq n.$$

(3) *For every locally finite open covering* $\{U_\gamma|\gamma<\tau\}$ *there exists a closed covering* $\mathcal{F}=\{F_\gamma|\gamma<\tau\}$ *of* R *such that*

$$F_\gamma\subset U_\gamma \text{ for every } \gamma<\tau,\ \mathrm{ord}\,\mathcal{F}\leqq n+1.$$

(4) *For every locally finite open covering* $\{U_\gamma|\gamma<\tau\}$ *of* R *there exists an open*

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covering $\mathfrak{V}=\{V_\gamma|\gamma<\tau\}$ *of* R *such that*

$$V_\gamma\subset U_\gamma,\ \text{ord}\,\mathfrak{V}\leqq n+1.$$

Proof. (1) implies (2): We denote the binary covering $\{U_\gamma, R-F_\gamma\}$ by $\mathcal{U}_\gamma$. Since $\wedge\{U_\gamma|\gamma<\tau\}$ is a locally finite open covering by virtue of the first part of the proof in Proposition II.5.B of [2], we can easily see that the statement holds. (2) implies (3): Let $\{U_\gamma|\gamma<\tau\}$ be a given locally finite open covering of R. In view of the normality of R there exists a closed covering $\{G_\gamma|\gamma<\tau\}$ such that $G_\gamma\subset U_\gamma$ for every $\gamma<\tau$. From the hypothesis, we can construct an open covering $\{V_\gamma|\gamma<\tau\}$ satisfying $G_\gamma\subset V_\gamma\subset\bar{V}_\gamma\subset U_\gamma$, ord $\{B(V_\gamma)|\gamma<\tau\}\leqq n$ for every $\gamma<\tau$. Let $F_0=\bar{V}_0$, $F_\gamma=\bar{V}_\gamma-\cup\{V_\alpha|\alpha<\gamma\}$ for $\gamma<\tau$. By virtue of Lemma 1, $\mathcal{F}=\{F_\gamma|\gamma<\tau\}$ is the desired covering. (3) implies (4) and (4) implies (1): In view of Proposition II.7.C of [2] and Dowker's Theorem, it is clear.

Let a space R be the sum of two closed sets F_1 and F_2. Let f_1 and f_2 be continuous mappings of F_1 and F_2 into S^n, respectively. If $\dim\{p|f_1(p)\neq f_2(p),\ p\in F_1\cap F_2\}\leqq n-1$, then f_1 can be continuously extended over R (Proposition III.3.C of [2]). We can generalize this proposition as follows.

THEOREM 2. *Let a metric space* R *be the sum of finite number of closed subsets* F_i, $i=0, 1, \cdots, k$, *and* f_i *continuous mappings of* F_i *into* S^n. *If*

$$\dim\{p|g_j(p)\neq f_{j+1}(p),\ p\in F_{j+1}\cap(F_0\cup F_1\cup\cdots\cup F_j)\}\leqq n-1$$

for any mapping g_j, $j=0, 1,\cdots, k-1$, *of* $F_0\cup F_1\cup\cdots\cup F_j$ *into* S^n, *then* f_0 *can be continuously extended over* R.

Proof. We shall carry out the proof by Proposition III.3.C of [2]. In case $j=0$, since f_0 and f_1 are continuous mappings of F_0 and F_1 into S^n, respectively, and $\dim\{p|\ g_0(p)\neq f_1(p),\ p\in F_1\cap F_0\}\leqq n-1$, then putting $g_0=f_0$ we have $\dim\{p|f_0(p)\neq f_1(p),\ p\in F_1\cap F_0\}\leqq n-1$. Hence there exists a continuous extension h_1 of f_0 over $F_0\cup F_1$. In case $j=1$, we can easily see that there exists a continuous extension h_2 of h_1 over $F_0\cup F_1\cup F_2$. By repeating this process, we have a continuous extension h_k of h_{k-1} over R. It is the desired extension of f_0.

COROLLARY. *Let a space* R *be the sum of finite number of closed subsets* F_i, $i=0, 1,\cdots, k$, *and* f_i *continuous mappings of* F_i *into* S^n. *If*

$$\dim F_i\leqq n-1,$$

then f_0 *can be continuously extended over* R.

3. Dimension of non-metrizable spaces. The purpose of this section is ready to extend the theory of dimension in metric spaces to somewhat more general spaces. Throughout this section we consider only T_2-spaces.

LEMMA 2. *If for any open covering* $\mathcal{U}=\{U_i|i=1, 2, \cdots, k\}$ *of a normal space* R *there exists an open covering* $\{V_i|i=1, 2, \cdots, k\}$ *such that*

$$\bar{V}_i\subset U_i,\ \dim B(V_i)\leqq n-1,\ i=1, 2, \cdots, k,$$

then R *has covering dimension* $\leqq n$.

Now letting $B=\bigcup_{i=1}^{k} B(V_i)$, then B is a closed set with $\dim B\leqq n-1$ by the general sum theorem. Therefore this Lemma can be easily proved by virtue of the process of the proof in Proposition VII.2.A of [2].

We prove the following proposition by applying the concept of the strong inductive dimension. It does throw a dim relation between covering dimension of a space R and strong inductive dimension of each members of an open covering.

PROPOSITION 1. *Let* $\mathfrak{U}=\{U_i|i=1, 2, \cdots, k\}$ *be any open covering of a normal space* R. *If* $\operatorname{Ind} U_i \leqq n$, *then* $\dim R\leqq n$.

Proof. Let $\{G_i|i=1, 2, \cdots, k\}$ be an open covering of R such that $\bar{G}_i\subset U_i$. Since R is normal, there exists an open covering $\{V_i| i=1, 2, \cdots, k\}$ such that $\bar{V}_i\subset G_i$. Thus $\bar{V}_i\cap\bar{G}_i$ and $(R-G_i)\cap\bar{G}_i$ are disjoint closed sets of $\bar{G}_i$. Since $\operatorname{Ind}\bar{G}_i\leqq n$, there exists an open set W_i of $\bar{G}_i$ such that $\bar{V}_i\subset W_i\subset G_i$, $\operatorname{Ind} B_{\bar{G}_i}(W_i)\leqq n-1$. Since $\dim B(W_i)\leqq \operatorname{Ind} B(W_i)\leqq n-1$ and $\operatorname{Ind} B_{\bar{G}_i}(W_i)=\operatorname{Ind} B(W_i)$, it follows from Lemma 2 that $\dim R\leqq n$.

Now we consider a finite open covering of a normal space R. Let $\{G_i|i=1, 2, \cdots, k\}$ be a closed covering of R such that $G_i\subset U_i$, $i=1, 2, \cdots, k$. By virtue of Theorem VII. 1 of [2] we can obtain an open covering $\{W_i|i=1, 2, \cdots, k\}$ satisfying $G_i\subset W_i\subset\bar{W}_i\subset U_i$, ord $\{B(W_i)| i=1, 2, \cdots, k\}\leqq n$. We define $F_i=\bar{W}_i-\bigcup_{j=1}^{i-1} W_j$, then we have $\operatorname{ord}\{F_i| i=1, 2, \cdots, k)\}\leqq n+1$. Therefore we can obtain the following.

PROPOSITION 2. *In a normal space* R *the followings are equivalent.*

(1) $\dim R\leqq n$.

(2) *For every open covering* $\{U_i|i=1, 2, \cdots, k\}$ *there exists a closed covering* $\mathfrak{F}=\{F_i|i=1, 2, \cdots, k\}$ *of* R *such that*

$$F_i\subset U_i,\ i=1, 2, \cdots, k$$
$$\operatorname{ord}\mathfrak{F}\leqq n+1.$$

(3) *For every open covering* $\{U_i|i=1, 2, \cdots, k\}$ *there exists an open covering* $\mathfrak{V}=\{V_i|i=1, 2, \cdots, k\}$ *of* R *such that*

$$V_i\subset U_i,\ i=1, 2, \cdots, k$$
$$\operatorname{ord}\mathfrak{V}\leqq n+1.$$

PROPOSITION 3. *Let* G_l, $l=0, 1, \cdots, k$, *be closed sets of a normal space* R *such that* $\dim G_l\leqq n-l$. *Let* $\{F_i|i=1, 2, \cdots k\}$ *be a closed collection and* $\{U_i| i=1,2,\cdots, k\}$ *an open collection such that*

$$F_i\subset U_i,\ i=1, 2, \cdots, k.$$

then there exists an open collection

$$\mathfrak{V}=\{V_i|i=1, 2, \cdots, k\}$$

such that

$$F_i\subset V_i\subset\bar{V}_i\subset U_i$$
$$\operatorname{ord}_p B(\mathfrak{V})\leqq n-l \textit{ for every } p\in G_l.$$

This can be proved by Proposition VII.1.B. of [2]. In case $l=0$ and $G_0=R$, we have Theorem VII.1 of [2].

Next we consider dimension of a hereditarily paracompact space R. Suppose $\{F_\gamma|\gamma<\tau\}$ is a locally finite closed covering of a paracompact space R. Then we can construct a locally finite open covering $\{P_\gamma|\gamma<\tau\}$ such that $F_\gamma\subset P_\gamma$ for every $\gamma<\tau$. Therefore from the sum theorem for strong inductive dimension we have the following.

LEMMA 3. *Let* $\{F_\gamma|\gamma<\tau\}$ *be a locally finite closed covering of a hereditarily* *paracompact space R such that*

$$\text{Ind } F_\gamma \leqq n \quad \textit{for every } \gamma<\tau.$$

Then $\text{Ind } R \leqq n.$

LEMMA 4. *Let* $\{F_\alpha | \alpha<\mu\}$ *be a countable closed covering of a hereditarily paracompact space R such that*

$$\text{Ind } F_\alpha \leqq n \quad \textit{for every } \alpha<\mu.$$

Then $\text{Ind } R \leqq n.$

This Lemma is obtained by a similar argument preceding Lemma 3. Here we denote by N the set of natural numbers containing zero and by ω the ordinal number of N, then μ means $\omega+1$. Combining Lemma 4 with Lemma 3 we obtain

PROPOSITION 4. *Let* $\{F_\gamma | \gamma<\tau\}$ *be a locally countable closed covering of a hereditarily paracompact space R such that*

$$\text{Ind } F_\gamma \leqq n \quad \textit{for every } \gamma<\tau,$$

then $\text{Ind } R \leqq n.$

Now we state a necessary and sufficient condition in order that a hereditarily paracompact space have strong inductive dimension $\leqq n$.

THEOREM 3. *A hereditarily paracompact space R has strong inductive dimension* $\leqq n$ *if and only if for every locally finite open collection* $\{U_\alpha | \alpha\in\Gamma\}$ *and closed collection* $\{F_\alpha | \alpha\in\Gamma\}$ of *R satisfying*

$$F_\alpha\subset U_\alpha \quad \textit{for every } \alpha\in\Gamma$$

there exists an open collection $\{V_\alpha | \alpha\in\Gamma\}$ *such that*

$$F_\alpha\subset V_\alpha\subset \bar{V}_\alpha\subset U_\alpha \quad \textit{for every } \alpha\in\Gamma,$$

$$\text{Ind } B(V_\alpha) \leqq n-1.$$

Proof. Necessity: Since F_α and $R-U_\alpha$ are disjoint closed sets of R, from the definition of strong inductive dimension it is clear.

Sufficiency: Let F and G be disjoint closed sets of R. Since R is paracompact, there exists a locally finite open covering $\mathcal{U}=\{U_\alpha | \alpha\in\Gamma\}$ of R such that $S(G,\mathcal{U})\cap F=\phi$. By virtue of the normality of R, there exists a closed covering $\mathcal{F}=\{F_\alpha | \alpha | \alpha\in\Gamma\}$ satisfying $F_\alpha\subset U_\alpha$ for every $\alpha\in\Gamma$. By the hypothesis we have an open covering $\mathfrak{B}=\{V_\alpha | \alpha\in\Gamma\}$ such thch $F_\alpha\subset V_\alpha\subset\bar{V}_\alpha\subset U_\alpha$, $\text{Ind } B(V_\alpha) \leqq n-1$ for every $\alpha\in\Gamma$. Let $B=\cup \{B(V_\alpha) | \alpha\in\Gamma\}$. Then by Lemma 3, B is a closed set with $\text{Ind } B \leqq n-1$. Let $V=S(G,\mathfrak{B})$ and $\mathfrak{B}=\{V_\alpha | \alpha\in \Gamma\}$, then $B(V) \subset B$, since $\mathfrak{B}$ is locally finite in R. It follows from $\text{Ind } B \leqq n-1$ that $\text{Ind } B(V) \leqq n-1$. Thus V is an open set of R such that $G\subset V\subset R-F$. Therefore

$$\text{Ind } R \leqq n.$$

COROLLARY. *A hereditarily paracompact space R has strong inductive dimension* $\leqq n$ *if and only if for any locally finite open covering* $\{U_\alpha | \alpha\in\Gamma\}$ *of R there exists an open covering* $\{V_\alpha | \alpha\in\Gamma\}$ *of R such that*

$$\bar{V}_\alpha\subset U_\alpha \quad \textit{for every } \alpha\in\Gamma,$$

$$\text{Ind } B(V_\alpha) \leqq n-1.$$

LEMMA 5. *Let* $\mathfrak{B}=\{V_\gamma | \gamma\in\Gamma\}$ *be an open covering of a topological space R such that*

$$\text{ord } B(\mathfrak{B}) \leqq n.$$

Let $\mathfrak{B}_\gamma=\{B(V_\gamma)\cap V_\alpha|\alpha\in\Gamma\}$ *for a fixed* γ, *then* $\mathfrak{B}_\gamma$ *is an open covering of* $B(V_\gamma)$ *such that*

$$\text{ord }\{B_{B(V_\gamma)}(H_\alpha)|H_\alpha\in\mathfrak{B}_\gamma\}\leqq n-1 \text{ where } H_\alpha=B(V_\gamma)\cap V_\alpha.$$

Proof. Let p be a point of $B(V_\gamma)$. If n members of $\{B_{B(V_\gamma)}(H_\alpha) \mid H_\alpha\in\mathfrak{B}_\gamma\}$ contain p, this leads us to ord $B(\mathfrak{B})\leqq n+1$. It is a contradiction.

The following proposition will be of some interest to see its covering order in the boundary of an open set.

PROPOSITION 5. *Let* $\mathfrak{U}=\{U_\gamma<\tau\}$ *be a locally finite open covering and* $\mathfrak{B}=\{V_\gamma|\gamma<\tau\}$ *an open covering of a hereditarily paracompact space R such that*

$$\bar{V}_\gamma\subset U_\gamma \quad \text{for every } \gamma<\tau$$

$$\text{ord }\{B(V_\gamma)|\gamma<\tau\}\leqq n.$$

Then there exists an open covering $W=\{W_\alpha|\alpha<\tau\}$ *of* $B(V_\gamma)$ *such that*

$$\text{ord } W\leqq n,$$

$$W_\alpha\subset U_\alpha\cap B(V_\gamma) \quad \text{for every } \alpha<\tau.$$

Proof. Let $\mathfrak{B}_\gamma=\{B(V_\gamma)\cap V_\alpha|\alpha<\tau\}$ for a fixed γ. By virtue of Lemma 5, $\mathfrak{B}_\gamma$ is an open covering of $B(V_\gamma)$ such that ord$\{B_{B(V_\gamma)}(H_\alpha)|H_\alpha\in\mathfrak{B}_\gamma\}\leqq n-1$. Let $F_0=\bar{H}_0^{B(V_\gamma)}$, and $F_\alpha=\bar{H}_\alpha^{B(V_\gamma)}-\cup\{H_\beta|\beta<\alpha\}$ for $\alpha<\tau$. From Lemma 1, $\mathcal{F}=\{F_\alpha|\alpha<\tau\}$ is a closed covering of $B(V_\gamma)$ such that ord $\mathcal{F}\leqq n$. On the other hand, $\{B(V_\gamma)\cap U_\alpha|\alpha<\tau\}$ is a locally finite open covering of $B(V_\gamma)$ such that $F_\alpha\subset B(V_\gamma)\cap U_\alpha$. By virtue of Proposition II.7. G of [2] we can construct the desired set W where $\bar{H}_\alpha^{B(V_\gamma)}$ is the closure of $\bar{H}_\alpha$ in the subspace $B(V_\gamma)$.

References

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