Degenerate Cases of the Einstein's Connection in the $*g^{\lambda\nu}$-Unified Field Theory-II. The Tensor $*U_{\nu\lambda\mu}$

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1. Introduction. This paper is a direct continuation of [3], in which we gave the singular solutions $S_{\omega\mu\nu}$ of the Einstein's equation for two degenerate cases and the necessary and sufficient conditions for the existence of at least one singular solution $S_{\omega\mu\nu}$ in the Einstein's $*g^{\lambda\nu}$-unified field theory. The purpose of the present paper is to give the corresponding singular solutions $*U_{\nu\lambda\mu}$ in holonomic and non-holonomic form and to derive the conditions for the existence of at least one singular solution $*U_{\nu\lambda\mu}$. This topic will be investigated for all possible indices of inertia.

2. Preliminary results. The notations and results employed in the present paper are essentially those of [1] and [3], and we begin with a brief recapitulation of the main results given there.

REMARK. All the considerations in the present paper are dealt for all possible indices of inertia. Furthermore, we restrict our discussions to two degenerate cases among three possible cases, namely the degenerate case of $*D=0$ of the first class (i.e., $*K=1/3$) and that of the second class (i.e., $*K=1/2$).

AGREEMENT. In our subsequent considerations we shall use the following Roman indices: $i, j, k=1, 2, 3, 4$; $a, b, c=1, 2$; $e, f, g=3, 4$. The indices a, b, c, e, f, g do not obey the usual summation convention.

A. In our $*g^{\lambda\nu}$-unified field theory whose differential geometrical structure is based on ([1], p. 1323)

$$D_\omega *g^{\lambda\mu}=-2S_{\omega\alpha}{}^{\mu} *g^{\lambda\alpha}, \tag{2.1a}$$

the connection $\Gamma^{\nu}_{\lambda\mu}$ must be of the form

$$\Gamma^{\nu}_{\lambda\mu}=*\{{}^{\nu}_{\lambda\mu}\}+S_{\lambda\mu}{}^{\nu}+*U^{\nu}{}_{\lambda\mu}, \tag{2.1b}$$

where

$$*U_{\nu\lambda\mu}\overset{\text{def}}{=}\overset{[10]0}{S}_{\lambda\mu\nu}+2\overset{[10]0}{S}_{\nu(\lambda\mu)}. \tag{2.2a}$$

Here $*\{{}^{\nu}_{\lambda\mu}\}$ are the Christoffel symbols defined by $*h_{\lambda\mu}$ in the usual way. In each of the following cases the tensor $*U_{\nu\lambda\mu}$ are found to be:

(*For the case $*D=0$ of the first class*)

$$(1+*K)(3*K-1)*U_{\nu\lambda\mu}=\frac{1}{2}(3*K-1)*\overset{[10]0}{K}_{\lambda\mu\nu}-*K*\overset{001}{K}_{\nu(\lambda\mu)}+*\overset{111}{K}_{\nu(\lambda\mu)}+(*K-1)*\overset{(10)0}{K}_{\nu(\lambda\mu)}. \tag{2.2b}$$

(*For the second class*)

$$(1-4*K^2)(2*U_{\nu\lambda\mu}+*\overset{[01]0}{K}_{\lambda\mu\nu}-2*\overset{(10)0}{K}_{\nu(\lambda\mu)}) =(2*K-1)*\overset{[21]0}{K}_{\lambda\mu\nu}+*\overset{[02]1}{K}_{\lambda\mu\nu}+2*K*\overset{[01]2}{K}_{\lambda\mu\nu}+*\overset{[21]2}{K}_{\lambda\mu\nu}-2\{2*K*\overset{(10)2}{K}_{\nu(\lambda\mu)}+ \tag{2.2c}$$

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$$+{}^*\overset{(20)1}{K}_{\nu(\lambda\mu)}+{}^*\overset{111}{K}_{\nu(\lambda\mu)}\}^{(*)}.$$

B. For each of the degenerate cases we have ([3], p.146)

(*For the case* $^*D=0$ *of the first class*)

(2.3) $$\underset{1}{^*\lambda}=-\underset{2}{^*\lambda}=i\sqrt{\frac{1}{3}},\quad \underset{3}{^*\lambda}=-\underset{4}{\lambda}=i\sqrt{\frac{1}{3}}$$

(*For the second class*)

(2.4) $$\underset{1}{^*\lambda}=-\underset{2}{^*\lambda}=i,\quad \underset{3}{^*\lambda}=-\underset{4}{^*\lambda}=0$$

(*For the case* $^*D=0$ *of the first class and for the second class*)

(2.5) $$^*K_{efg}={}^*K_{abc}=0,\quad {}^*K_{fab}={}^*K_{f(ab)},\quad {}^*K_{aef}={}^*K_{a(ef)}.$$

The non-holonomic components of the singular solutions $S_{\omega\mu\nu}$ of (2.1)a are given by:

(*For the case* $^*D=0$ *of the first class*)

a) S_{aff} and S_{faa} are arbitrary when $a+f$ is even;

b) $S_{aff}=\frac{3}{8}\,{}^*K_{aff},\ S_{faa}=\frac{3}{8}{}^*K_{faa}$ when $a+f$ is odd;

c) $S_{abc}=S_{efg}=0;$

(2.6) d) $$\begin{cases} S_{abf}=\frac{3}{8}\left\{{}^*K_{abf}+\frac{1}{3}(-1)^{b+f}\,{}^*K_{fab}\right\}, & (a\neq b)\\ S_{efa}=\frac{3}{8}\left\{{}^*K_{efa}+\frac{1}{3}(-1)^{a+f}\,{}^*K_{afe}\right\}, & (e\neq f)\end{cases}$$

e) $S_{fab}=\frac{1}{2}{}^*K_{fab},\ (a\neq b);\ S_{afe}=\frac{1}{2}{}^*K_{afe},\ (e\neq f).$

(*For the second class*)

a) S_{faa} are arbitrary; b) $S_{abf}=\frac{1}{4}{}^*K_{abf}$;

(2.7) c) $S_{afg}=\frac{1}{2}{}^*K_{afg}$; d) $S_{afb}=\frac{1}{2}{}^*K_{afb}\ (a\neq b)$;

e) $S_{efa}=\frac{1}{2}{}^*K_{efa}$; f) $S_{abc}=S_{efg}=0.$

3. Singular solutions $^*U_{\nu\lambda\mu}$ **for the case** $^*D=0$ **of the first class.** In this section we restrict our considerations to the degenerate case $^*D=0$ of the first class, namely $^*K=1/3$.

THEOREM (3.1). *A necessary and sufficient condition that* (2.2)a *admits at least one solution* $^*U_{\nu\lambda\mu}$ *is*

(3.1)a $$-\frac{1}{3}{}^*\overset{001}{K}_{\nu(\lambda\mu)}+{}^*\overset{111}{K}_{\nu(\lambda\mu)}-\frac{2}{3}{}^*\overset{(10)0}{K}_{\nu(\lambda\mu)}=0,$$

which is equivalent to its non-holonomic form

(3.1)b $$^*K_{faa}={}^*K_{aff}=0 \qquad \textit{when } a+f \textit{ is even.}$$

Proof. Since $^*K=1/3$, the condition (3.1)a is evident from (2.2)b. Using the relation ${}^{(p)*}k_{\underset{x}{}}{}^{i}=(\underset{x}{^*\lambda})^p\delta_x{}^i$, (3.1)a is reduced to its non-holonomic form

(3.2) $$I^*K_{x(yz)}=0 \qquad \text{where } I=(\underset{x}{^*\lambda}+\underset{y}{^*\lambda}+\underset{z}{^*\lambda}-3\underset{x}{^*\lambda}\underset{y}{^*\lambda}\underset{z}{^*\lambda})$$

(*) Note that the result (7.3)b, obtained in [1], p.1323, is wrong. The correct solution is as given in (2.2)c above.

We see that I is identically zero for all x,y,z except for the following two cases:

Case 1. $x=f,\ y=a,\ z=a;\ x=a,\ y=f,\ z=a;$ or $x=a,\ y=a,\ z=f$ when $a+f$ is even, in which case, using (2.3) and (2.5), (3.2) is reduced to $*K_{faa}=0$;

Case 2. $x=a,\ y=f,\ z=f;\ x=f,\ y=a,\ z=f;$ or $x=f,\ y=f,\ z=a$ when $a+f$ is even, in which case, using (2.5) and (2.3) again, (3.2) is reduced to $*K_{aff}=0$.

Hence (3.1)b follows.

REMARK. Note that the condition (3.1)b is identical to the existence condition for at least one $S_{\lambda\mu\nu}$ (See [3], p. 147). This fact is not surprising in view of (2.2)a.

THEOREM (3.2). *When the conditions* (3.1) *are satisfied, the tensor* $*U_{\nu\lambda\mu}$ *has the following non-holonomic components*:

a) $*U_{abi}=*U_{efi}=*U_{iii}=0,\ (a\neq b,\ e\neq f)$;

b) $*U_{aff},\ *U_{aaf},\ *U_{faa}$, *and* $*U_{faf}$ *are arbitrary when* $a+f$ *is even*;

$$\text{(3.3) c)}\ \begin{cases} *U_{aff}=*U_{faa}=0 \\ *U_{aaf}=\frac{3}{8}\underset{a}{*\lambda}\ *K_{afa}, & \text{when } a+f \text{ is odd}: \\ *U_{faf}=\frac{3}{8}\underset{f}{*\lambda}\ *K_{faf} \end{cases}$$

$$\text{d)}\ \begin{cases} *U_{afe}=\frac{3}{8}(\underset{a}{*\lambda}*K_{afe}+\underset{f}{*\lambda}*K_{fea}), & (e\neq f) \\ *U_{fab}=\frac{3}{8}(\underset{f}{*\lambda}*K_{fab}+\underset{a}{*\lambda}*K_{abf}), & (a\neq b). \end{cases}$$

Proof. In general the non-holonomic components of $*U_{\nu\lambda\mu}$ may be obtained from (2.2)a as follows:

$$*U_{xyz}=\frac{1}{2}\{S_{xyz}(\underset{x}{*\lambda}+\underset{y}{*\lambda})+S_{xzy}(\underset{x}{*\lambda}+\underset{z}{*\lambda})+S_{yzx}(\underset{y}{*\lambda}-\underset{z}{*\lambda})\}. \tag{3.4}$$

Hence (3.3) follows from (3.4), using (2.3), (2.5) and (2.6). For example,

$$\begin{aligned} *U_{fab}&=\frac{1}{2}\left\{S_{fab}(\underset{f}{*\lambda}+\underset{a}{*\lambda})+S_{fba}(\underset{f}{*\lambda}+\underset{b}{*\lambda})+S_{abf}(\underset{a}{*\lambda}-\underset{b}{*\lambda})\right\} \\ &=\frac{1}{2}\left\{\frac{1}{2}*K_{fab}(\underset{f}{*\lambda}+\underset{a}{*\lambda})+\frac{1}{2}*K_{fba}(\underset{f}{*\lambda}+\underset{b}{*\lambda})+\frac{3}{4}\underset{a}{*\lambda}\left[*K_{abf}+\frac{1}{3}(-1)^{b+f}*K_{fab}\right]\right\} \\ &=\left\{\frac{1}{2}\underset{f}{*\lambda}+\frac{1}{8}(-1)^{b+f}\underset{a}{*\lambda}\right\}*K_{fab}+\frac{3}{8}\underset{a}{*\lambda}*K_{abf} \\ &=\left(\frac{1}{2}\underset{f}{*\lambda}-\frac{1}{8}\underset{f}{*\lambda}\right)*K_{fab}+\frac{3}{8}\underset{a}{*\lambda}*K_{abf} \\ &=\frac{3}{8}(\underset{f}{*\lambda}*K_{fab}+\underset{a}{*\lambda}*K_{abf}). \end{aligned}$$

Note that the last two steps of the above calculation must be carried out for both cases that $a+f$ is even and that $a+f$ is odd.

THEOREM (3.3). *When the conditions* (3.1) *are satisfied, the solution* $*U_{\nu\lambda\mu}$ *of* (2.2)a *may be given by*

$$*U_{\nu\lambda\mu}=\frac{1}{2}\overset{[10]0}{*K_{\lambda\mu\nu}}+\overset{(10)0}{*K_{\nu(\lambda\mu)}}+D_{\nu\lambda\mu}, \tag{3.5}$$

where $D_{\nu\lambda\mu}$ *is a tensor symmetric in the last two indices such that in non-holonomic frame*

a) $D_{abi}=D_{efi}=D_{iii}=0$, $(a\neq b,\ e\neq f)$;

b) $D_{aff}, D_{aaf}, D_{faa}$, *and* D_{ffa} *are arbitrary when* $a+f$ *is even*;

$$(3.6)\ \text{c)}\ \begin{cases} D_{aff}=D_{faa}=0 \\ D_{aaf}=\frac{1}{8}{}^*\underset{a}{\lambda}{}^*K_{faa} \\ D_{ffa}=\frac{1}{8}{}^*\underset{f}{\lambda}{}^*K_{aff} \end{cases} \quad \text{when } a+f \text{ is odd:}$$

$$\text{d)}\ \begin{cases} D_{afe}=\frac{1}{8}({}^*\underset{f}{\lambda}{}^*K_{efa}-{}^*\underset{a}{\lambda}{}^*K_{afe}),\ (e\neq f) \\ D_{fab}=\frac{1}{8}({}^*\underset{a}{\lambda}{}^*K_{baf}-{}^*\underset{f}{\lambda}{}^*K_{fab}),\ (a\neq b). \end{cases}$$

Proof. In view of (2.2)a, (4.20)([1], p. 1311) and (4.18)([1], p. 1311) it is clear that the holonomic components of ${}^*U_{\nu\lambda\mu}$ can be put in the form (3.5). Since the non-holonomic components of the tensor $D_{\nu\lambda\mu}$ can be written

$$D_{xyz}={}^*U_{xyz}-\frac{1}{4}\{({}^*\underset{y}{\lambda}-{}^*\underset{z}{\lambda}){}^*K_{yzx}+({}^*\underset{x}{\lambda}+{}^*\underset{y}{\lambda}){}^*K_{xyz}+({}^*\underset{x}{\lambda}+{}^*\underset{z}{\lambda}){}^*K_{xzy}\},$$

(3.6) may be easily obtained by using (2.3), (2.5), and (3.3). For example, (3.6)d can be derived as follows:

$$\begin{aligned} D_{afe}&={}^*U_{afe}-\frac{1}{4}\{({}^*\underset{f}{\lambda}-{}^*\underset{e}{\lambda}){}^*K_{fea}+({}^*\underset{a}{\lambda}+{}^*\underset{f}{\lambda}){}^*K_{afe}+({}^*\underset{a}{\lambda}+{}^*\underset{e}{\lambda}){}^*K_{aef}\} \\ &=\frac{3}{8}({}^*\underset{a}{\lambda}{}^*K_{afe}+{}^*\underset{f}{\lambda}{}^*K_{fea})-\frac{1}{2}\left({}^*\underset{e}{\lambda}{}^*K_{fea}+\frac{1}{2}{}^*\underset{a}{\lambda}{}^*K_{afe}\right) \\ &=\frac{1}{8}({}^*\underset{f}{\lambda}{}^*K_{efa}-{}^*\underset{a}{\lambda}{}^*K_{afe}). \end{aligned}$$

4. Singular solutions ${}^*U_{\nu\lambda\mu}$ for the second class. In this section we restrict our considerations to the degenerate case of the second class, namely ${}^*K=1/2$.

THEOREM (4.1). *A necessary and sufficient condition that* (2.2)a *admits at least one solution* ${}^*U_{\nu\lambda\mu}$ *is*

$$(4.1)\text{a}\qquad {}^*\overset{(02)1}{K}_{\lambda\mu\nu}+{}^*\overset{(01)2}{K}_{\lambda\mu\nu}+{}^*\overset{(21)2}{K}_{\lambda\mu\nu}-2\{{}^*\overset{(10)2}{K}_{\nu(\lambda\mu)}+{}^*\overset{(20)1}{K}_{\nu(\lambda\mu)}+{}^*\overset{(11)}{K}_{\nu(\lambda\mu)}\}=0,$$

which is equivalent to its non-holonomic form

$$(4.1)\text{b}\qquad {}^*K_{faa}=0.$$

Proof. Since ${}^*K=1/2$, the condition (4.1)a is evident from (2.2)c. Using the relation ${}^{(p)*}\underset{r}{k}_x{}^i=({}^*\lambda)^p\delta_x{}^i$, (4.1)a is reduced to to its non-holonomic form

$$(4.2)\qquad \begin{aligned} &{}^*\underset{z}{\lambda}({}^*\underset{x}{\lambda}{}^2{}^*\underset{y}{\lambda}-{}^*\underset{x}{\lambda}{}^*\underset{y}{\lambda}{}^2+{}^*\underset{z}{\lambda}{}^2-{}^*\underset{x}{\lambda}{}^2+{}^*\underset{y}{\lambda}{}^*\underset{z}{\lambda}-{}^*\underset{x}{\lambda}{}^*\underset{z}{\lambda}){}^*K_{xyz}+ \\ &+{}^*\underset{y}{\lambda}({}^*\underset{y}{\lambda}{}^*\underset{z}{\lambda}+{}^*\underset{x}{\lambda}{}^*\underset{y}{\lambda}+{}^*\underset{z}{\lambda}{}^2+{}^*\underset{x}{\lambda}{}^2+2{}^*\underset{x}{\lambda}{}^*\underset{z}{\lambda}){}^*K_{zxy} \\ &+{}^*\underset{x}{\lambda}({}^*\underset{x}{\lambda}{}^*\underset{z}{\lambda}+{}^*\underset{x}{\lambda}{}^*\underset{y}{\lambda}+{}^*\underset{z}{\lambda}{}^2+{}^*\underset{y}{\lambda}{}^2+2{}^*\underset{y}{\lambda}{}^*\underset{z}{\lambda}){}^*K_{zyx}=0. \end{aligned}$$

We see that (4.2) is identically zero for all x, y, z except for the cases

$x=f,\ y=a,\ z=a$; $x=a,\ y=f,\ z=a$; and $x=a,\ y=a,\ z=f$.

In these cases, using (2.4) and (2.5), (4.2) is reduced to (4.1)b.

REMARK. In this case again, we note that the condition (4.1)b is identical to the corresponding existence condition for at least one singular solution $S_{\omega\mu\nu}$ (see [3], p. 149).

THEOREM (4.2). *When the condition* (4.1) *is satisfied, the tensor* ${}^*U_{\nu\lambda\mu}$ *has the* *following non-holonomic components*:

a) $*U_{abi}=*U_{fgi}=*U_{aaa}=0,\ (a\neq b)$;

(4.3) b) $*U_{aaf}$ *and* U_{faa} *are arbitrary*;

c) $\begin{cases} *U_{afg}=\frac{1}{2}*\lambda*K_{afg}; \\ *U_{fab}=\frac{1}{4}\underset{a}{*\lambda}*K_{abf},\ (a\neq b). \end{cases}$

Proof Using the similar method to the proof of Theorem (3.2), (4.3) follows from (3.4) in virtue of (2.4), (2.5), and (2.7).

THEOREM. (4.3). *When the conditions* (4.1) *are satisfied, the solution* $*U_{\nu\lambda\mu}$ *of* (2.2)a *may be given by*

$$*U_{\nu\lambda\mu}=\frac{1}{2}*\overset{(10)0}{K}_{\lambda\mu\nu}+*\overset{(10)0}{K}_{\nu(\lambda\mu)}+E_{\nu\lambda\mu}, \tag{4.4}$$

where $E_{\nu\lambda\mu}$ *is a tensor symmetric in the last two indices such that in non-holonomic frame*

a) $*E_{abi}=*E_{fgi}=*E_{aaa}=*E_{afg}=0,\ (a\neq b)$;

(4.5) b) $*E_{aaf}$ *and* $*E_{faa}$ *are arbitrary*;

c) $*E_{fab}=\frac{1}{4}\underset{b}{*\lambda}*K_{abf},\ (a\neq b)$.

Proof. In view of (2.2)a, (4.20)([1], p.1311) and (4.18)([1], p.1311) it is clear that the holonomic components of $*U_{\nu\lambda\mu}$ can be put in the form (4.4). Since the non-holonomic components of the tensor $E_{\nu\lambda\mu}$ can be written

$$E_{xyz}=*U_{xyz}-\frac{1}{4}\{(\underset{y}{*\lambda}-\underset{z}{*\lambda})*K_{yzx}+(\underset{x}{*\lambda}+\underset{y}{*\lambda})*K_{xyz}+(\underset{x}{*\lambda}+\underset{z}{*\lambda})*K_{xzy}\},$$

(4.5) is obtained from (2.4), (2.5), and (4.3), using the similar method to the proof of Theorem (3.3).

5. The connection $\Gamma_{\lambda\mu}^{\nu}$. Now that we have obtained the singular solutions $S_{\omega\mu\nu}$ in our previous paper [3] and the singular solutions $*U_{\nu\lambda\mu}$ in the present paper in both holonomic and non-holonomic form for two degenerate cases, it is possible for us to determine the respective degenerate connection $\Gamma_{\lambda\mu}^{\nu}$ by simply substituting for $S_{\lambda\mu}{}^{\nu}$ and $*U^{\nu}{}_{\lambda\mu}$ into (2.1)b.

References

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