J. Korean Math. Soc. 7(1970), pp 15—18

A Note on the Complexification of a Ring

YOUNG L. PARK

The main objective of this note is to describe a relationship between the structure space of a ring and the corresponding space of its complexification. Also as byproducts it is shown that the maximal ring of quotients $Q(C(X))$ of the ring $C(X)$ of all complex-valued continuous functions on X does not have any complex ideal if X is a separable metric space without isolated points, and the structure space of $Q(C(X))$ is the projective cover [4] of the Stone-Čech compactification βX of a completely regular Hausdorff space X.

1. Complexification. The symbol A will be used throughout this section to represent a commutative ring with unit e. Consider the cartesian product $A\times A$, and define the operation $+$ and $\cdot$ on $A\times A$ by $(a,b)+(c,d)=(a+c,\ b+d)$ and $(a,b)\cdot(c,d)=(ac-bd,\ ad+bc)$. Put $A_C=(A\times A,+,\cdot)$. Then it is readily verified that A_C is a commutative ring with unit $(e,0)$ and the mapping $a\longrightarrow(a,0)$ is an isomorphism of A into A_C. We call the ring A_C the complexification of A. Also it is easy to check that if I is an ideal in A then its complexification I_C is also an ideal in A_C. For a subset S of A_C we put

$$r(S)=\{a\in A\,|\,(a,b)\in S \text{ for some } b\in A\},$$
$$i(S)=\{b\in A\,|\,(a,b)\in S \text{ for some } a\in A\}.$$

Then clearly $S\subset r(S)\times i(S)$. We prove following lemmas.

LEMMA 1. *If J is an ideal in A_C, then $r(J)=i(J)$ $(\equiv I)$ and I is an ideal in A.*

Proof. Let $x\in r(J)$. Then there exists $x'\in A$ such that $(x,x')\in J$. Since $e\in A$, $(0,\ e)(x,x')=(-ex',\ ex)=(-x',\ x)\in J$; hence $x\in i(J)$, *i. e.*, $r(J)\subseteq i(J)$. Also $i(J)\subseteq r(J)$ can be shown in the same way. To show I an ideal, let $x,\ y\in I$. Then there exists $x',\ y'\in A$ such that $(x,\ x'),\ (y,\ y')\in J$. Hence $(x+y,\ x'+y')\in J$, *i.e.*, $x+y\in I$. Since J is an ideal in A_C and $(x,\ 0)\in A_C$, $(x,\ 0)(y,\ y')=(xy,\ xy')\in J$; and hence $xy\in$ I. Finally let $a\in A$ and $x\in$ I. Then there exists $x'\in A$ such that $(x,\ x')\in$ I, and $(a,\ 0)(x,\ x')=(ax,\ ax')=(ax,\ ax')\in J$, i.e., $ax\in I$.

The proof of the following Lemma is straightforward.

LEMMA 2. *For an ideal I in A, $(A/I)_C\cong A_C/I_C$.*

Received by the editors March 2, 1970.

REMARK. We remarked that, for a subset $J\subset A_C$, $J\subset r(J)\times i(J)$. In general $J\neq r(J)\times i(J)$ even though J is an ideal in A_C. For instance, consider $A=Z_2$ and $J=\{(0,\ 0),\ (1,\ 1)\}$, then clearly J is an ideal A_C. But $r(J)=i(J)=\{0,1\}$ and $J\neq r(J)\times i(J)$. This means that for arbitrary ideal J in A_C there may not exist an ideal I in A such that $J=I_C$. However, we have the following situation.

DEFINITION 1. *A ring A is said to be weakly convex if all $e+a^2$, $a\in A$ are invertible in A.*

LEMMA 3. *Let A be a weakly convex ring. If M' is a maximal ideal in A_C, then there exists a maximal ideal M in A such that $M'=M_C$.*

Proof Put $M=r(M')$. Then M is an ideal in A. We show M is proper. Suppose $e\in M$, then $(e,\ b)\in M'$ for some $b\in A$, and hence $(e(e+b^2)^{-1},\ -b(e+b^2)^{-1})(e,b)=(e,0)\in M'$, a contradiction; hence $e\notin M$. Clearly $M'\subset M\times M$. Take any $(a,b)\in M\times M$. Suppose $(a,b)\notin M'$. Then there exists $(x,y)\in A_C$ such that $(e,0)-(a,b)(x,\ y)\in M'$ since $(a,\ b)+M'\neq 0$ in A_C/M'. Then $(e-(ax-by),\ -(ay+bx))\in M'$. Hence $e-(ax-by)\in M$ where $a,\ b\in M$ and $x,\ y\in A$. This implies that $e\in M$, a contradiction. Thus $M'=M_C$. Now we show that M is maximal in A. Let $a\notin M$. Then$(a+\mathrm{M},o')\neq 0$ in $(A/M)_C$ where o' denotes the zero in A/M. Since $(A/M)_C$ is a field, hence there exists $x,\ y\in A$ such that $(a+M,\ o')(x+M,\ y+M)=(e+M,\ o')$. Thus $((e-ax)+M,\ -ay+M)=(o',\ o')$, and hence $e-ax\in M$. This concludes that M is a maximal ideal in A.

LEMMA 4. *If K is a formally real field, then its complexification K_C is a field.*

Proof. Clearly K_C is a commutative ring with unit $(e,0)$. Let $(a,b)\neq 0$ in K_C; then $a\neq 0$ or $b\neq 0$, or both $a,b\neq 0$ in K. Hence $a^2+b^2\neq 0$ in K. Clearly $(a(a^2+b^2)^{-1}$, $-b(a^2+b^2)^{-1})\in K_c$ and hence $(a(a^2+b^2)^{-1},\ -b(a^2+b^2)^{-1})(a,b)=(e,0)$, i.e., each nonzero element of K_C is invertible.

REMARK. In general, K_C needs not be a field even if K is a field. For instance, if K is the field of complex numbers, then K_C is not a field since the element $(1,i)$ does not have an inverse element in K_C. Also consider the ring Z_2, then Z_2 is a field, but the complexification $(Z_2)_C$ is not a field since $(1,1)$ does not have an inverse element in $(Z_2)_C$. As a matter of fact $(Z_2)_C$ is not even an integral domain.

NOTATION. By $\Omega(A)$ we denote the structure space of A, i.e., the maximal ideal space of A endowed with the Stone topology.

THEOREM. *Let A be a weakly convex ring such that, for each maximal ideal M,* *the quotient field A/M is formally real. Then* $\Omega(A)$ is homemorphic to $\Omega(A_C)$.

Proof. The mapping $\theta : \Omega(A) \longrightarrow \Omega(A_C)$ defined by $\theta(M)=M_C$ is clearly one to one, and in virtue of the Lemma 3 the mapping θ is onto. To show that θ is a homeomorphism, it suffices to show that $\theta(\Omega(ab))=\Omega_C((a, b))$ where $\Omega(a)$, $a \in A$ and $\Omega_C((a, b))$, $(a, b) \in A_C$ are the basic open sets of $\Omega(A)$ and $\Omega(A_C)$ respectively. Ths follows from the fact that $ab \notin M$ iff $a \notin M$ and $b \notin M$, and iff $(a,b) \notin M_C=\theta(M)$.

2. Applications. Let X denote a completely regular Hausdorff space and ϑ a filter base of dense subsets of X. Denote by $C_\vartheta(X)$ the ring of all complex valued functions f on X which have continuous restriction $f|D$ to some member D of ϑ and by $Z_\vartheta(X)$ the ideal of $C_\vartheta(X)$ consisting of all functions f with $f|D=0$ for some D in ϑ. Put $Q_\vartheta(X)=C_\vartheta(X)/Z_\vartheta(X)$. It is known in [1] that if ϑ is the set of all dense open subsets of X then $Q_\vartheta(X)$ is the ring of fractions of $C(X)$, i.e., $Q_\vartheta(X)$ is the maximal ring of quotients of $C(X)$. The corresponding ring for the real-valued functions will be denoted by $Q_\vartheta(X, R)$, i.e., $Q_\vartheta(X,R)=C_\vartheta(X,R)/Z_\vartheta(X,R)$. It is evident that, for $u \in Q_\vartheta(X)$ with $u=f+Z_\vartheta(X)$, the mapping $u \longrightarrow (Re(f)+Z_\vartheta(X, R), Im(f)+Z_\vartheta(X, R))$ is an isomorphism of $Q_\vartheta(X)$ onto $Q_\vartheta(X,R)_C$. Also we note that $Q_\vartheta(X,R)$ is weakly convex.

DEFINITION 2. *A maximal ideal M of a ring A is said to be complex (resp. real) if the quotient field A/M is isomorphic to the field of complex (resp. real) numbers. A ring is said to be totaly uncomplex (resp. totally unreal) if it does not have any complex (resp. real) ideal.*

In virtue of Lemma 2 and 3, the Corollary 1 is an immediate consequence.

COROLLARY 1. *If a ring A is weakly convex, then A_C is totally uncomplex iff A is totally unreal.*

COROLLARY 2. *If $Q_\vartheta(X)$ is totally uncomplex, then $\cap\vartheta=\phi$, and the converse holds, provided each member of ϑ is realcompact.*

Proof. If $Q_\vartheta(X,R)$ is totally unreal, then $\cap\vartheta=\phi$ [6], and the converse holds if each member of ϑ is realcompact.

COROLLARY 3. *Let X be a separable realcompact space without isolated point such that every closed subset is G_δ-set; then the maximal ring of quotients of $C(X)$ is totally uncomplex.*

COROLLARY 4. *For a separable metric space X without isolated point, the maximal ring of quotients of $C(X)$ is totally uncomplex.*

LEMMA 5. *For a filter base ϑ of dense subsets of X, the quotient field $Q_\vartheta(X, R)/M$ is real-closed for each maximal ideal M.*

Proof. Essentially the same as in [3, Theorem 13.4].

COROLLARY 5. *The structure space of the maximal ring of quotients of $C(X)$ is the projective cover of βX.*

Proof. Let ϑ be the set of all dense open subsets of X. From Lemma 5, for each maximal ideal M, $Q_\vartheta(X, R)/M$ is formally real. Hence from Theorem 1, we have $\Omega(Q(C(X))\cong\Omega(Q_\vartheta(X))\cong\Omega(Q_\vartheta(X, R))\cong\Omega(Q(C(X, R)))$. It is proved in [5] that $\Omega(Q(C(X, R)))$ is the projective cover of βX.

Biblography

1. B. Banaschewski, *Maximal Rings of Quotients of Semi-simple Commutative Rings.* Arch. Math., 16(1965), 414—420.
2. N. Fine, L. Gillman and J. Lambek, *Rings of Quotients of rings of Functions.* McGill Univ. Press, Montreal, 1965.
3. L. Gillman and M. Jerison, *Rings of continuous Functions.* Princeton, 1960.
4. A.M. Gleason. *Projective Topological Space.* Ill. J. Math., 2(1958), 482—489.
5. Y.L. Park, *On the Projective Cover of the Stone Čech Compactification of a Completely Regular Hausdorff Space.* Can. Math. Bull. Vol. 12, No. 3, (1969), 327—331.
6. Y.L. Park, *On Quotient-like Extensions of $C(X)$* Can. Math. Bull. Vol. 12, No. 6, (1969).

Laurentian University