A CORRECT SYSTEM OF AXIOMS FOR A SYMMETRIC GENERALIZED TOPOLOGICAL GROUP

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In this paper we introduce the concept of a symmetric generalized topological group and show its relationship to the concept of a symmetric generalized uniform space (introduced by the author in [2]).

Throughout this paper $(G, \cdot)$ will denote a group with identity ε. $\mathscr{T}$ will denote a topology on G, and $\mathscr{N}$ will denote an open base at ε. $A^{-1}=\{a^{-1} \mid a \in A\}$, $AB=\{ab \mid a \in A,\ b \in B\}$ where A and B are subsets of G.

(1.1) DEFINITION. $(G, \cdot, \mathscr{T})$ is a *symmetric generalized topological group* iff the following axioms are satisfied:

(A.1) For every $x \in G$ $\{ xN \mid N \in \mathscr{N} \}$ is an open base at x.

(A.2) For every $N \in \mathscr{N}$ $N=N^{-1}$.

(1.2) REMARK. If we require that the mapping $f:(x,y)\to xy$ of $(G\times G)$ onto G be continuous in each variable separately, then $\{xN \mid N \in \mathscr{N}\}$ and $\{Nx \mid N \in \mathscr{N}\}$ are bases at x for every $x \in G$. If we require that the mapping $g:x\to x^{-1}$ of G onto G be continuous, then for every $N \in \mathscr{N}$ $N^{-1} \in \mathscr{N}$. This latter fact implies that for every $N \in \mathscr{N}$ $(N \cap N^{-1}) \in \mathscr{N}$. But $(N\cap N^{-1})^{-1}=(N\cap N^{-1})$.

(1.3) REMARK. It is easily shown that if F is closed, P is open, and A is an arbitrary subset of G and if x is an arbitrary point in G, then xF, F^{-1}, are closed and xP, P^{-1}, and AP are open subsets of G where $(G, \cdot, \mathscr{T})$is a symmetric generalized topological group.

(1.4) THEOREM. *If $(G, \cdot, \mathscr{T})$ is a symmetric generalized topological group, then* $\bar{A}=\bigcap \{AN \mid N \in \mathscr{N}\}$.

PROOF. Let $y \in \bar{A}$ and $N \in \mathscr{N}$. Then $yN^{-1} \cap A \neq \phi$; so that $y \in AN$. Conversely, suppose $y \in AN$ for every $N \in \mathscr{N}$. Then $y \in AN^{-1}$for every $N \in \mathscr{N}$; consequently, $yN \cap A \neq \phi$ for every $N \in \mathscr{N}$; so that $y \in \bar{A}$.

(1.5) THEOREM. *Let $(G, \cdot, \mathscr{T})$ be a symmetric generalized topological group. Let F be a closed and C a compact subset of G such that $F \cap C=\phi$. Then there exists $N \in \mathscr{N}$ such that $FN \cap CN=\phi$.*

PROOF. Let $M \in \mathscr{N}$. Let $F_M = \overline{FMM^{-1}}$. Then by Theorem (1.4) $F_M = \cap \{FM \cdot M^{-1}W \mid W \in \mathscr{N}\} = F$. Hence $F_M \cap C = \phi$ for each $M \in \mathscr{N}$; so that $\{G - F_M \mid M \in \mathscr{N}\}$ is an open covering of C. Hence there is a finite subfamily F_{M_i} $(1 \leq i \leq n)$ such that

$$\left(\bigcap_{i=1}^{n} F_{M_i}\right) \cap C = \phi .$$

Let $N = \cap M_i$. It is easily shown that

$$NN^{-1} = \cap M_i N^{-1} \subset \cap M_i M_i^{-1}$$

so that

$$FNN^{-1} \subset \cap FM_i M_i^{-1} .$$

By taking closures we see that

$$\overline{FNN^{-1}} \subset F_N \subset \cap F_{M_i} .$$

Hence $FNN^{-1} \cap C = \phi$; so that $FN \cap CN = \phi$.

We now investigate the relationship between symmetric generalized topological groups and symmetric generalized uniform spaces.

(1.6) THEOREM. *Let $(G, \cdot, \mathscr{T})$ be a symmetric generalized topological group. For each $N \in \mathscr{N}$ let $U_N = \{(x, y) \mid x^{-1}y \in N\}$. Let $\mathscr{B}$ be the collection of all U_N. Then $\mathscr{B}$ is a base for a symmetric generalized uniformity, $\mathscr{U}(G)$, on G such that $\mathscr{T}(\mathscr{U}(G)) = \mathscr{T}$.*

NOTE. $AN = \cup \{xN \mid x \in A\}$; so that by (A.1) AN is open. But by hypothesis there exists $b \in AN \cap B$. Since AN is open, b is an interior point of AN; consequently, by (A.1) there exists $W \in \mathscr{N}$ such that $bW \subset AN$.

PROOF of THEOREM(1.6). Clearly, to show $\mathscr{B}$ is a base for some symmetric generalized uniformity $\mathscr{U}$ on G it is sufficient to show that for every $N \in \mathscr{N}$ and for all subsets A, B of G, if $U_M[A] \cap B \neq \phi$ for every $M \in \mathscr{N}$, then there exists $b \in B$ and there exists $W \in \mathscr{N}$ such that $U_W[b] \subset U_N[A]$. But since we have that $U_N[A] = \cup \{xN \mid x \in A\} = AN$ for all $N \in \mathscr{N}$ and for each subset A of G, this is an immediate consequence of the note above. It is clear that $\mathscr{T}(\mathscr{U}(G)) = \mathscr{T}$.

(1.8) COROLLARY. *If $(G, \cdot, \mathscr{T})$ is a symmetric generalized topological group and $\mathscr{N}$ has a least element, say N_0, then $N_0^2 \subset N$ for every $N \in \mathscr{N}$.*

PROOF. Clearly, for every $U \in \mathscr{U}(G)$ we have that $U_{N_0} \subset U$. Consequently, by lemma (2.32) in [2] $U_{N_0} \circ U_{N_0} \subset U$ for every $U \in \mathscr{U}(G)$. Hence if $(x, y) \in U_{N_0}$

and $(y, z) \epsilon U_{N_0}$, then $(x, z) \epsilon U_N$ for every $N \epsilon \mathscr{N}$. That is to say for every $N \epsilon \mathscr{N}$ if $x^{-1}y \epsilon N_0$ and $y^{-1}z \epsilon N_0$, then $x^{-1}z \epsilon N_0$. Let $p \epsilon N_0$ and $q \epsilon N_0$. Then p^{-1} is in N_0; so that $P^{-1}\varepsilon$ is in N_0 and $\varepsilon^{-1}q$ is in N_0. Hence $pq \epsilon N$. Thus $N_0^2 \subset N$.

(1.9) THEOREM. *If $(G, \cdot, \mathscr{T})$ is a locally compact symmetric generalized topological group, then $\mathscr{U}(G)$ is complete.*

PROOF. Let $\mathscr{F}$ be any filter in G that is weakly Cauchy with respect to $\mathscr{U}(G)$. Since $(G, \cdot, \mathscr{T})$ is locally compact, there exists a compact neighborhood $N \epsilon \mathscr{N}$, and since $\mathscr{F}$ is weakly cauchy with respect to $\mathscr{U}(G)$, there exists an $x_0 \epsilon G$ such that $U_N [x_0] = x_0 N \epsilon \mathscr{F}$. By (A.1) it is easily shown that xN_0 is compact. We now let $\mathscr{B} = \{E \mid E = F \cap x_0 N$ *for some* $F \epsilon \mathscr{F}\}$. It is easily shown that B is a base for a filter $\mathscr{F}_1$ in x_0N; but since x_0N is compact, $\mathscr{F}_1$ has a cluster point $x_1 \epsilon x_0N$; which clearly is a cluster point for $\mathscr{F}$. Hence $(G, \mathscr{U}(G))$ is complete.

(1.10) THEOREM. *If $(G, \cdot, \mathscr{T})$ is a locally compact, T_2, symmetric generalized topological group, then $(G, \cdot, \mathscr{T})$ is a topological group.*

For a proof of this rather deep result the reader is referred to [1].

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REFERENCES

[1] Ellis, R.: *Locally compact transformation groups.* Duke Math. J., 24, 119-125, 1957.
[2] Mozzochi, C. J.: *Symmetric generalized topological structures.* (publication pending).