REMARKS ON TOPOLOGICAL LATTICES

By Tae Ho Choe

In [2] Anderson has conjectured that if L is a locally compact connected topological lattice, then L is chainwise connected, i.e., any pair of two points x and y with $x \leq y$ can be contained in a closed connected chain in L. Anderson and Ward [3] have given an affirmative answer for this; they have proved it in the fashion of topological semi-lattices.

In this remark, we shall first give its another direct proof and using this, we shall prove that any locally compact connected topological lattice with 0 and 1 is an acyclic, i.e., $H^p(L)=0$ for all integers $p \geq 1$, where $H^*(\quad)$ denotes the Alexander-Kolmogoroff-Spanier cohomology group with a non-trivial additive coefficient group. We shall next give serveral equivalent conditions for a complete Boolean topological lattice to be iseomorphic (i.e., lattice-isomorphic and homeomorphic) to the Boolean topological lattice 2^X of all subsets of some set X, where 2 has the discrete topology. Finally we shall give an affirmative answer to problem 85 in [5]; Is every complete morphism (i.e., for arbitrary joins and meets) of complete lattices continuous in the interval topology?

THEOREM 1. *If L is a locally compact connected topological lattice, then L is chainwise connected.*

PROOF. For a given pair of comparable points a and b with $a \leq b$, the closed interval $M=[a,b]$ $(=a\vee(b\wedge L))$ is also a locally compact connected topological lattice in its relative topology.

Let C be the set of all points p in M such that there exists a compact connected chain $C(a,p)$ from a to p. It is easy to see that $a \leq q \leq p$ and $p \in C$ imply $q \in C$ (consider $q \wedge C(a,p)$).

We now show that C is open in M. For an element p of C choose neighborhoods U, V and W of p in M such that V is convex, W^* compact and $U \vee U \subset V \subset W^*$ (M is locally convex [1]). For an arbitrary element u in U we have the closed interval $N=[p, p\vee u]$ which is contained in V. And N is a compact connected topological lattice in its relative topology. It is well known that a compact connected topological lattice is chain-wise connected.

Therefore there exists a compact connected chain $C(p, p\vee u)$. Seeing $C(a, p)\cup C(p, p\vee u)$, we have $p\vee u \in C$. Thus $u \in C$ and hence C is open in M. Suppose $M\backslash C\neq\square$. Then for an element t of $M\backslash C$, again choose neighborhoods U, V and W of t such as the above. If $U\cap C\neq\square$, then for $s\in U\cap C$ we have $[s, s\vee t]\subset V\subset W^*$. By the reasoning used before it follows that $t\in C$. This is a contradiction. Therefore $U\subset M\backslash C$. Thus C is a non-void closed and open subset of M, and hence $C=M$.

COROLLARY 2. *If L is a locally compact connected topological lattice with* 0 *and* 1, *then L is an acyclic.*

PROOF. Let I be a compact connected chain in L from 0 to 1. Let i be the indentity mapping of L to itself and let g be the constant mapping of L into L defined by $g(x)=0$ for all x in L. Considering a mapping Φ from $L\times I$ into L defined by $\Phi(x, c)=x\wedge c$ for an element $x\in L$ and an elemnt $c\in I$ we have that i is a null homotopy. By the homotopy axion of the cohomology, the induced mappings i^* and g^* are the same. Since i^* is an isomorphism and $H^p(\{0\})=0$ for all integers $p\geq 1$, we have $H^p(L)=0$ for all integers $q\geq 1$.

THEOREM 3. *In a compact topological lattice, distributivity implies infinite distributivity.*

PROOF. Let L be a compact distributive topological lattice. We show that for an element x of L and any non-void subset B of L, $x\wedge(\vee B)=\vee(x\wedge B)$, where $\vee B=\sup B$ and $x\wedge B=\{x\wedge b \mid b\in B\}$. Let Γ be the set of all finite subsets of B. Setting $z_G=\vee G$ for each $G\in\Gamma$, and taking the inclusion relation as the directing relation on Γ, we have that the net $\{z_G \mid G\in\Gamma\}$ is monotone increasing and $\vee\{z_G \mid G\in\Gamma\}=\vee B$. Since L is compact, the net $\{z_G \mid G\in\Gamma\}$ converges to its supremum $\vee\{z_G \mid G\in\Gamma\}$, [7]. Hence the net $\{x\wedge z_G \mid G\in\Gamma\}$ converges to $x\wedge(\vee\{z_G \mid G\in\Gamma\})$. Since $x\wedge z_G=\vee(x\wedge G)$, again setting $u_G=\vee(x\wedge G)$, we have that the net $\{u_G \mid G\in\Gamma\}$ converges to $\vee\{u_G \mid G\in\Gamma\}$ $(=\vee(x\wedge B))$. Hence $x\wedge(\vee B)=\vee(x\wedge B)$ as required.

By the interval topology of a lattice L, denoted by $I(L)$, we mean the topology defined by taking the closed intervals. $\{a\wedge L,\ a\vee L \mid a\in L\}$ as a subbase for the closed sets.

For a net $\{x_\alpha \mid \alpha\in D\}$ in a complete lattice L, if lim sup $\{x_\alpha \mid \alpha\in D\}=$ lim inf $\{x_\alpha \mid \alpha\in D\}=x$, we say that the net $\{x_\alpha\}$ order converges to x. We define a subset M of L to be closed in the order topology of L, denoted by $0(L)$ if and only if no net in M order converges to a point outside of M.

By a complete subset C of a lattice L we shall mean a non-void subset C of L such that for each non-void subset S of C, S possesses both a sup S and an inf S in L, and, furthermore, both sup S and inf S are in C. For a lattice L, the smallest topology for L in which the complete subsets of L are closed is called the complete topology of L, denoted by $C(L)$. It is well known that $C(L)\subset 0(L)$ [8], and if L is complete, then $I(L)\subset C(L)$.

THEOREM 4. *If L is a complete Boolean lattice then the following are equivalent:*
(i) *$I(L)$ is Hausdorff.*
(ii) *$C(L)$ is Hausdorff.*
(iii) *The meet (or join) operation is continuous on $0(L)$ and $0(L)$ is compact.*
(iv) *L is atomic, and hence $L\cong 2^X$ for some set X.*

PROOF. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). By Corollary 4 in [8], $0(L)$ is compact. Suppose $\{(x_\alpha, y_\alpha)\mid \alpha\in D\}$ converges to (x,y) in $(L, 0(L))\times(L, 0(L))$. If z is a cluster point of the net $\{x_\alpha\wedge y_\alpha\mid\alpha\in D\}$, then there is a subnet $\{x_\alpha\wedge y_\alpha\mid\alpha\in D'\}$ of $\{x_\alpha\wedge y_\alpha\}$ which converges to z. Clearly, $\{x_\alpha\mid\alpha\in D'\}$ and $\{y_\alpha\mid\alpha\in D'\}$ are subnets of $\{x_\alpha\mid\alpha\in D\}$ and $\{y_\alpha\mid\alpha\in D\}$, respectively, and contain subnets $\{x_\beta\mid\beta\in D''\}$ and $\{y_\beta\mid\beta\in D''\}$ which order converge to x and y, respectively ([8], Theorem 3). Therefore $\{x_\beta\wedge y_\beta\mid\beta\in D''\}$ order converges to $x\wedge y$ [4], and hence $z=x\wedge y$. It follows that $\{x_\alpha\wedge y_\alpha\mid\alpha\in D\}$ converges to $x\wedge y$.

(iii) $\Rightarrow$ (iv). It is not difficult to see that the unary operation of complementation in L is always continuous in $0(L)$. Now suppose that the meet-operation is continuous in $0(L)$. Using De Morgan's formulas we can easily see that the join-operation is also continuous. Hence L is a topological group under the symmetric difference operation so that L is a regular space in $0(L)$. Since $0(L)$ is always T_1, $0(L)$ is Hausdorff. It follows that $(L, 0(L))$ is compact Boolean topological lattice. Hence L is iseomorphic with 2^X for some set X, where 2 has the discrete topology, [6].

(iv) $\Rightarrow$ (i) is known from [9].

A mapping of complete lattice into a complete lattice is a complete morphism for arbitrary joins and meets iff the mapping preserves arbitrary joins and meets.

THEOREM 5. *Every complete morphism for arbitrary joins and meets of complete lattices is continuous in the interval topology.*

The proof of theorem 5 is immediately from a known result [4] that if L is a complete lattice and $\{x_a | a \in A\}$, a net in L, then $\{x_a | a \in A\}$ converges to a point x in the interval topology iff $\vee_C \wedge \{x_c : c \in C\} \leq x \leq \wedge_C \vee \{x_c : c \in C\}$, where C denotes an arbitrary cofinal subset of A.

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