NOTES ON CLOSED HYPERSURFACES IN A RIEMANNIAN SPACE WITH CERTAIN DIFFERENTIAL EQUATION

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1. Introduction

Let M be an n-dimensional oritentable Riemannian manifold covered by a system of coordinate neighbourhoods (ξ^h) and g_{ji}, ∇_i, K_{kjih}, K_{ji}, and K, the positive definite fundamental tensor, the operator of covariant differentiation with respect to Christoffel symbols $\left\{ {h \atop ji} \right\}$ formed with g_{ji}, the curvature tensor, the Ricci tensor, and the curvature scalar of M respectively, where here and in the following the indices $h, i, j, \cdots$ run over the range $1, 2, \cdots, n$. Recently K. Yano [1] proved

THEOREM. *Let M be an orientable Riemannian manifold of dimension n which admits a non-constant scalar field v such that*

$$\nabla_j \nabla_i v = f(v) g_{ji},$$

where f is a differentiable function of v and S a closed orientable hypersurface in M such that

(i) *its first mean curvature is constant,*

(ii) $[K_{ji} + (n-1)f'(v)g_{ji}]C^jC^i \geqslant 0$ *on S, where C^h is the unit normal to S,*

(iii) *the inner product $C^i \nabla_i v$ has fixed sign on S.*

Then every point of S is umbilical.

To obtain a generalization of avobe theorem, we assume in this paper the existence of a non-constant scalar function which satisfies similar partial differential equation. While, under an arbitrary conformal transformation $\bar{g}_{jk} = v^2 g_{jk}$ any geodesic circle will be transformed into a geodesic circle if and only if the function v satisfies the relation

$$\nabla_j \nabla_i v = \sigma g_{ji} + v_j v_i$$

where $$v_j = \frac{\partial \log v}{\partial \xi^j}$$

and such a conformal transformation will be called *concircular* [4].

2. Formulas in M admitting any Scalar Field

We consider a closed orientable hypersurface S in a Riemannian manifold M whose local parametric equations are

$$\xi^h=\xi^h(\eta^a),$$

η^a being parameters on S, where here and in the following the indices $a, b, c, \cdots$ run over the range $1, 2, \cdots, n-1$.

If we put

$$B_b^h=\partial_b\xi^h, \qquad \partial_b=\partial/\eta^b,$$

then B_b^h are $n-1$ linearly independent vectors tangent to S and the first fundamental tensor of S is given by

$$g_{cb}=g_{ji}B_c^jB_b^i.$$

We assume that $n-1$ vectors $B_1^h, B_2^h, \cdots, B_{n-1}^h$ give the positive orientation on S and we denote by C^h the unit normal vector to S such that

$$B_1^h, B_2^h, \cdots, B_{n-1}^h, C^h$$

give the positive orientation in M.

Denoting by ∇_c the operator of van der Waerden-Bortolotti covariant differentiation along S we have the following equations of Gauss and of Weingarten:

$$(2\cdot1) \qquad \nabla_cB_b^h=h_{cb}\,C^h,$$

$$(2\cdot2) \qquad \nabla_cC^h=-h_c^aB_a^h,$$

where h_{cb} is the second fundamental tensor of S and $h_c^{\ a}=h_{cb}g^{ba}$. We also obtain the equations of Gauss and those of Codazzi in the form

$$K_{kjih}B_d^kB_c^jB_b^iB_a^h=K_{dcba}-(h_{da}h_{cb}-h_{ca}h_{db}),$$

$$K_{kjih}B_d^kB_c^jB_b^iC^h=\nabla_dh_{cb}-\nabla_c h_{db},$$

where K_{dcba} is the curvature tensor of the hypersurface S. From the equations of Codazzi, we have, by a transvection with g^{cb},

$$(2\cdot3) \qquad K_{kh}B_d^k\,C^h=\nabla_d\,h_c^{\ c}-\nabla_c\,h_d^{\ c}.$$

We now assume that the Riemannian manifold M admits a non-constant scalar field v such that

$$(2\cdot4) \qquad \nabla_jv_i=f(v)g_{ji}+v_jv_i, \qquad v_i=\nabla_iv$$

where $f(v)$ is a differentiable function of v.

The condition (2·4) is a formal generalization of a concircular transformation in a Riemannian space. We shall call such a transformation a *f-concircular transformation.*

We put

$$v^h = B_a^{\ h} v^a + \alpha C^h \tag{2·5}$$

on the hypersurface S. From (2·4) we obtain by transvection with $B_c^{\ j} B_b^{\ i}$

$$\nabla_c v_b = f(v) g_{cb} + \alpha h_{cb} + v_b v_c, \tag{2·6}$$

from which

$$\Delta v = (n-1) f(v) + \alpha h_c^{\ c} + v^a v_a, \tag{2·7}$$

where Δ is the Laplacian operator on S: $\Delta = g^{cb} \nabla_c \nabla_b$.

From (2.4), we also obtain by transvection with $B_b^{\ j} C^i$

$$\nabla_b \alpha = -h_b^{\ a} v_a + \alpha v_b \tag{2.8}$$

On the other hand, substituting (2.4) into the Ricci identity

$$\nabla_k \nabla_j v_i - \nabla_j \nabla_k v_i = -K_{kji}^{\ \ \ h} v_h,$$

we find that

$$-K_{kji}^{\ \ \ h} v_h = f'(v)(v_k g_{ji} - v_j g_{ki}) + f(v)(g_{ki} v_j - g_{ji} v_k),$$

from which

$$K_{ij} v^j = -(n-1)(f'(v) - f(v)) v_i,$$

and consequently

$$K_{ji} v^j C^i = -(n-1)\alpha(f'(v) - f(v)),$$

which can also be written as

$$K_{ji}(B_c^{\ j} v^c + \alpha C^j) C^i = -(n-1)\alpha(f'(v) - f(v)),$$

or, by virtue of (2.3),

$$(\nabla_c h_b^{\ b} - \nabla_b h_c^{\ b}) v^c + \alpha K_{ji} C^j C^i = -(n-1)\alpha(f'(v) - f(v)),$$

that is

$$\alpha K_{ji} C^j C^i + (n-1)(f'(v) - f(v))\alpha + v^c \nabla_c h_b^{\ b} - \nabla_b (h_c^{\ b} v^c) + h_{cb} v^b v^c + f(v) h_b^{\ b} + \alpha h_c^{\ b} h_b^{\ c} = 0 \tag{2.9}$$

by virtue of (2.6).

We now assume that the hypersurface S is closed and the first mean curvature of S is constant.

Applying Green's theorem to (2.7) and (2.9), we obtain

(2.10) $$(n-1)\int_S f(v)dS+\int_S \alpha h_c^{\ c} dS+\int_S v^a v_a dS=0$$

and

(2.11) $$\int_S [\alpha K_{ji}C^j C^i+(n-1)\alpha(f'(v)-f(v))+f(v)h_b^{\ b}+\alpha h_c^{\ b} h_b^{\ c}+h_{cb}v^c v^b]\,dS=0$$

respectively, where dS denotes the surface element of S.

Eliminating $\int_S f(v)ds$ from these two equations, we find that

(2.12) $$\int_S \Big[\alpha(K_{ji}+(n-1)\alpha(f'(v)-f(v))+\alpha\Big(h^{cb}-\frac{1}{n-1}h_t^{\ t}g^{cb}\Big)\Big(h_{cb}-\frac{1}{n-1}h_s^{\ s}g_{cb}\Big) + \Big(h_{cb}-\frac{1}{n-1}h_t^{\ t}g_{cb}\Big)v^c v^b\Big]dS=0$$

On the other hand, from(2.6) and(2.8) we have

$$\nabla_c\nabla_b\,\alpha=-(\nabla_c h_{ba})\,v^a-h_{ba}(\alpha h_c^{\ a}+v_c v^a+f(v)\delta_c^{\ a}) +(-h_{ca}v^a+\alpha v_c)v_b+\alpha(\alpha h_{cb}+v_c v_b+f(v)g_{cb}),$$

from which

(2.13) $$\Delta\alpha=-\nabla_c(v^a h_a^{\ c})-h_{ab}v^a v^b+\alpha^2 h_c^{\ c}+2\alpha v^a v_a+f(v)(n-1)\alpha.$$

Applying Green's theorem to (2·13) we find that

(2·14) $$\int_S [h_{cb}v^b v^c-\alpha^2 h_c^{\ c}-2\alpha v^a v_a-(n-1)\alpha f(v)]\,dS=0.$$

From (2·10), (2·11) and (2·14) we have

(2·15) $$\int_S \alpha\Big[(K_{ji}+(n-1)f'(v)g_{ji})C^j C^i+\Big(h^{cb}-\frac{1}{n-1}h_t^{\ t}g^{cb}\Big)\Big(h_{cb}-\frac{1}{n-1}h_s^{\ s}g_{cb}\Big)\Big]dS +\int_S\Big[2\alpha-\frac{1}{n-1}h_c^{\ c}\Big)v^a v_a+\alpha^2 h_c^{\ c}\Big]dS=0.$$

3. Results

From (2·12) and (2·15) we have immediately the following

THEOREM 1. *Let M be an orientable Riemannian manifold of dimension n which admits a proper f-concircular transformation and S a closed c?ientable hypersurface in M such that*

(i) *its first mean curature is constant,*

(ii) $K_{ji}+(n-1)(f'(v)-f(v))g_{ji}]C^j C^i \geqq 0$ *on S, where C^h is the unit normal to S,*

(iii) $(h_{cb}-\frac{1}{n-1}h_a^{\ a}g_{cb})v^c v^b$, $C^i\nabla_i v$ *have the same fixed sign on S,*

or (i) *and* (ii)′ $[K_{ji}+(n-1)f'(v)g_{ji}]C^j C^i\geqq 0$ *on S* (iii)′ $C^i\nabla_i v\geqq\frac{1}{2(n-1)}h_a^{\ a}\geqq 0$ *on. S.*

Then every points of S is umbilical.

THEOREM 2. *Let M be an orientable Riemannian manifold of dimension n which admits a proper f-concircular transformation and S a closed orientable hypersurface in M such that*

(i) *its first mean curvature is constant,*

(ii) $C^i\nabla_i v$ *is positive on S,*

(iii) $\{K_{ji}+[(n-1)(f'(v)-f(v))-\frac{1}{4\alpha^2}v^a v_a]\,g_{ji}\}C^jC^i\geqq 0$ *on S.*

Then v^h *is normal to S.*

PROOF. From (2·10) and (2·11) we get

$$\int_s \alpha\Big[K_{ji}C^jC^i+(n-1)(f'(v)-f(v))+h_c{}^b h_b{}^c-\frac{1}{n-1}\,h_c{}^c h_b{}^b-\frac{1}{\alpha(n-1)}\,h_c{}^c v^a v_a+\frac{1}{\alpha}\,h_{cb}v^c v^b\Big]ds=0,$$

by virtue of $C^i\nabla_i v=\alpha$.

or

$$\int_s \alpha\Big[\Big(K_{ji}+\Big\{(n-1)(f'(v)-f(v))-\frac{1}{4\alpha^2}\Big\}g_{ji}\Big)C^jC^i+\Big(h^{cb}-\frac{1}{n-1}h_t{}^t g^{cb}+\frac{1}{2\alpha}\,v^c v^b\Big)\Big(h_{cb}-\frac{1}{n-1}h_s{}^s g_{cb}+\frac{1}{2\alpha}\,v_b^c v_c^b\Big)\Big]dS=0.$$

Therefore $h_{cb}-\frac{1}{n-1}h_t{}^t g_{cb}+\frac{1}{2\alpha}v_b v_c=0$. Hence $v_c=0$.

These complete the proof.

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