A THEOREM ON HANKEL TRANSFORM

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1. Introduction

Let

$$H\{f(x)\}^{\gamma}=g(y)=\int_0^{\infty} f(x)\ J_{\gamma}(xy)\cdot(xy)^{\frac{1}{2}}\ dx,$$

the Hankel transform of order γ of $f(x)$, where y is a positive real variable. This form of the Hankel transform has the advantage of reducing to the Fourier sine or cosine tranform when $\gamma=\pm\frac{1}{2}$. Many authors regard

$$\int_0^{\infty} f(x)\ J_{\gamma}(xy)x\ dx \quad \text{or} \quad \int_0^{\infty} f(x)\ J_{\gamma}[2(xy)]^{\frac{1}{2}}\ dx$$

as the Hankel transform of order γ of $f(x)$. The Hankel transform is self reciprocal.

The object of this paper is to obtain a theorem involving chain of Hankel transform.

2. THEOREM. *Let* (i) $H[f(x)]^{\gamma}=g(y)$

$$\text{(ii)}\quad H\left[g\left(\frac{1}{x}\right)\right]^{\gamma}=\phi_1(y). \tag{2.1}$$

Then

$$H\left[x^{\frac{3}{2}}f\left(\frac{x^2}{4}\right)\right]^{2\gamma}=4y^{\frac{3}{2}}\phi_1(y^2), \tag{2.2}$$

provided $f(x)$, $x^{\frac{1}{4}}f(x)$, *and* $g\left(\frac{1}{x}\right)$ *are bounded and absolutely integrable in* $(0,\ \infty)$. Re $\gamma>-\frac{1}{2}$.

Further, let

$$H\left[x^{-\frac{7}{2}}\phi_1\left(\frac{1}{x^2}\right)\right]^{2\gamma}=\frac{1}{4}\phi_2(y), \tag{2.3}$$

$$H\left[x^{-\frac{3}{2}}\phi_2\left(\frac{1}{2x^2}\right)\right]^{4\gamma}=\frac{1}{2^{\frac{3}{2}}}\phi_3(y), \tag{2.4}$$

$$H\left[x^{-\frac{3}{2}}\phi_3\left(\frac{1}{2x^2}\right)\right]^{8\gamma}=\frac{1}{2^{\frac{5}{2}}}\phi_4(y), \tag{2.5}$$

....

$$H\Big[x^{-\frac{3}{2}}\phi_{n-1}\Big(\frac{1}{2x^2}\Big)\Big]^{2^{n-1}\gamma}=\frac{1}{2^{\left(2^{n-3}+\frac{1}{2}\right)}}\phi_n(y). \qquad (2.6)$$

Then

$$H\Big[x^{\left(2^{n-1}+\frac{1}{2}\right)}f\Big(\frac{x^{2^n}}{2^{2^n}}\Big)\Big]^{2^n\gamma}=\frac{2^{\left(2^{n-2}+\frac{1}{2}\right)}}{y^{\frac{1}{2}}}\phi_n\Big(\frac{y^2}{2}\Big), \qquad (2.7)$$

provided Re $n>1$, *n is an integer, and under the conditions mentioned above.*

PROOF. *Let* $\int_0^\infty f(x)J_\gamma(xy)(xy)^{\frac{1}{2}}dx=g(y)$.

Multiplying both sides by $y^{-\frac{5}{2}}J_\gamma\Big(\frac{a}{y}\Big)$ and integrating with respect to y between the limits $(0, \infty)$, we obtain

$$\int_0^\infty J_\gamma\Big(\frac{a}{y}\Big)\frac{dy}{y^{\frac{5}{2}}}\int_0^\infty J_\gamma(xy)(xy)^{\frac{1}{2}}f(x)dx=\int_0^\infty J_\gamma\Big(\frac{a}{y}\Big)g(y)\frac{dy}{y^{\frac{5}{2}}}.$$

On changing the order of integrations, which is justified by the conditions given in theorem, and evaluating the integral [3, p. 57] on the left hand side, we obtain (2.2), on using (2.1) on right hand side.

We obtain $x^{-\frac{7}{2}}\phi_1\Big(\frac{1}{x^2}\Big)$ from (2.2), Substituting it in (2.3), we have

$$\phi_2(y)=\int_0^\infty J_{2\gamma}(xy)(xy)^{\frac{1}{2}}\Big[\int_0^\infty t^2f\Big(\frac{t^2}{4}\Big)J_{2\gamma}\Big(\frac{t}{x}\Big)\frac{dt}{x^{\frac{5}{2}}}\Big]dx$$

On changing the order of integrations which is permissible by the conditions given in the theorem and evaluating the x-integral [3, p. 57], we obtain

$$H\Big[t^{\frac{5}{2}}f\Big(\frac{t^4}{2^4}\Big)\Big]^{4\gamma}=\frac{2^{\frac{3}{2}}}{y^{\frac{1}{2}}}\phi_2\Big(\frac{y^2}{2}\Big).$$

We get $x^{-\frac{3}{2}}\phi_2\Big(\frac{1}{2x^2}\Big)$. Substituting it in (2.4) and proceeding as before, we have

$$H\Big[t^{\frac{9}{2}}f\Big(\frac{t^8}{2^8}\Big)\Big]^{8\gamma}=\frac{2^{\frac{5}{2}}}{y^{\frac{1}{2}}}\phi_3\Big(\frac{y^2}{2}\Big).$$

Proceeding successively we assume the result (2.7).

Let $$H\Big[x^{-\frac{3}{2}}\phi_n\Big(\frac{1}{2x^2}\Big)\Big]^{2^n\gamma}=\frac{1}{2^{\left(2^{n-2}+\frac{1}{2}\right)}}\phi_{n+1}(y). \qquad (2.8)$$

We have $x^{-\frac{3}{2}}\phi_n\left(\frac{1}{2x^2}\right)$ from (2.7). Substituting it in (2.8), we obtain

$$\phi_{n+1}(y)=\int_0^\infty J_{2^n\gamma}(xy)(xy)^{\frac{1}{2}}\left[x^{-\frac{5}{2}}\int_0^\infty t^{(2^{n-1}+1)}f\left(\frac{t^{2^n}}{2^{2^n}}\right)J_{2^n\gamma}\left(\frac{t}{x}\right)dt\right]dx.$$

On changing the order of integrations and evaluating the x-integral as before, we obtain

$$H\left[t^{\left(2^n+\frac{1}{2}\right)}f\left(\frac{t^{2^{n+1}}}{2^{2^{n+1}}}\right)\right]^{(2^{n+1}\gamma)}=\frac{2^{\left(2^{n-1}+\frac{1}{2}\right)}}{y^{\frac{1}{2}}}\phi_{n+1}\left(\frac{y^2}{2}\right)$$

We thus find that if (2.7) is true for n, then it is also true for $(n+1)$. But we have seen that it is true for $n=2$, therefore it is true for $n=3$. Since the result is true for $n=3$, therefore it is true for $n=4$, and so on. Hence (2.7) is true for all positive integral values of n except one.

COROLLARY. (i) *Let* $\gamma=\frac{1}{2^{n+1}}$. *We obtain the Fourier sine transform of*

$$\left[x^{\left(2^{n-1}+\frac{1}{2}\right)}f\left(\frac{x^{2^n}}{2^{2^n}}\right)\right]=\frac{\pi^{\frac{1}{2}}\cdot 2^{(2^{n-2})}}{y^{\frac{1}{2}}}\phi_n\left(\frac{y^2}{2}\right).$$

(ii) *Let* $\gamma=-\frac{1}{2^{n+1}}$. *We obtain the Fourier cosine transform of*

$$x^{\left(2^{n-1}+\frac{1}{2}\right)}f\left(\frac{x^{2^n}}{2^{2^n}}\right)=\frac{\pi^{\frac{1}{2}}\cdot 2^{(2^{n-2})}}{y^{\frac{1}{2}}}\phi_n\left(\frac{y^2}{2}\right).$$

3. Application.

Let $$f(x)=x^{\left(2\rho-\frac{1}{2}\right)}G^{h,k}_{p,q}\left(\delta x^2\,\middle|\,\begin{matrix}a_1,\cdots\cdots,a_p\\ \beta_1,\cdots\cdots,\beta_q\end{matrix}\right)$$

$$\therefore\ g(y)=\frac{2^{2\rho}}{y^{\left(2\rho+\frac{1}{2}\right)}}G^{h,k+1}_{p+2,q}\left(\frac{4\delta}{y^2}\,\middle|\,\begin{matrix}\frac{1}{2}-\rho-\frac{\gamma}{2},\ a_1,\cdots\cdots,a_p,\frac{1}{2}-\rho+\frac{\gamma}{2}\\ \beta_1,\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\beta_q\end{matrix}\right),\quad [3,\ p.91],$$

$(p+q)<2(h+k)$, Re $\left(\beta_j+\rho+\frac{\gamma}{2}\right)>-\frac{1}{2}$, $j=1,\ 2,\ \cdots\cdots,\ h$,

Re $(a_j+\rho)<\frac{3}{4}$, $j=1,\cdots\cdots,k$, and $|\arg\delta|<\left(h+k-\frac{p}{2}-\frac{q}{2}\right)\pi$.

$$\therefore\ \phi_1(y)=\frac{2^{(4\rho+1)}}{y^{\left(2\rho+\frac{3}{2}\right)}}G^{h,k+2}_{p+4,q}\left(\frac{16\delta}{y^2}\,\middle|\,\begin{matrix}-\rho-\frac{\gamma}{2},-\rho-\frac{\gamma}{2}+\frac{1}{2},a_1,\cdots\cdots,a_p,-\rho+\frac{\gamma}{2},\frac{1}{2}-\rho+\frac{\gamma}{2}\\ B_1,\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\ B_q\end{matrix}\right),$$

[3, p. 91], $(p+q)<2(h+k)$, Re $(a_j+\rho)<\frac{1}{4}$, $j=1,\ \cdots\cdots\cdots\cdots,k$,

$|\arg\delta|<\left(h+k-\frac{p}{2}-\frac{q}{2}\right)\pi$, Re $\left(\beta_j+\rho+\frac{\gamma}{2}\right)>-\frac{1}{2}$, $j=1,\ \cdots,h$.

Then we get from (2.2)

$$H\left[x^{\left(4\rho+\frac{1}{2}\right)}G^{h,k}_{p,q}\left(\frac{\delta x^4}{2^4}\,\middle|\,\begin{matrix}a_1,\cdots\cdots,a_p\\ \beta_1,\cdots\cdots,\beta_q\end{matrix}\right)\right]^{2\gamma}$$

$$=\frac{2^{(8\rho+2)}}{y^{\left(4\rho+\frac{3}{2}\right)}}G^{h,k+2}_{p+4,q}\left(\frac{16\delta}{y^4}\,\middle|\,\begin{matrix}-\rho-\frac{\gamma}{2},\frac{1}{2}-\rho-\frac{\gamma}{2},a_1,\cdots,a_p,-\rho+\frac{\gamma}{2},-\rho+\frac{\gamma}{2}+\frac{1}{2}\\ \beta_1,\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\ \beta_q\end{matrix}\right),\quad(3.1)$$

$(p+q)<2(h+k)$, Re $(a_j+\rho)<\frac{1}{4}$, $j=1,\ 2,\cdots\cdots,\ k$, Re $\left(\beta_j+\rho+\frac{\gamma}{2}\right)>-\frac{1}{2}$,

$j=1,\ 2,\cdots\cdots,h$, and $|\arg\delta|<\left(h+k-\frac{p}{2}-\frac{q}{2}\right)\pi$.

We obtain $\phi_2(y)$ from (2.3), on using (3.1). Let $n=2$.

We obtain from (2.7)

$$H\left[x^{\left(8\rho+\frac{1}{2}\right)}G^{h,k}_{p,q}\left(\frac{\delta x^8}{2^8}\,\middle|\,\begin{matrix}a_1,\cdots\cdots,a_p\\ \beta_1,\cdots\cdots,\beta_q\end{matrix}\right)\right]^{4\gamma}=\frac{2^{(24\rho+3)}}{y^{\left(8\rho+\frac{3}{2}\right)}}G^{h,k+4}_{p+8,q}\left(\frac{2^{16}\delta}{y^8}\,\middle|\,\begin{matrix}-\rho-\frac{\gamma}{2},-\rho-\frac{\gamma}{2}+\frac{1,2,3^*}{4},a_1,\cdots\cdots,a_p,-\rho+\frac{\gamma}{2},-\rho+\frac{\gamma}{2}+\frac{1,2,3}{4}\\ \beta_1,\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\ \beta_q\end{matrix}\right),\quad(3.2)$$

$(p+q)<2(h+k)$, Re $(a_j+\rho)<\frac{1}{4}$, $j=1,\cdots\cdots k$, Re $\left(\beta_j+\rho+\frac{\gamma}{2}\right)>-\frac{1}{4}$,

$j=1,\cdots,\ h$ and $|\arg\delta|<\left(h+k-\frac{p}{2}-\frac{q}{2}\right)\pi$.

We have $\phi_3(y)$ from (2.4), on using (3.2). Let $n=3$. We obtain from (2.7)

$$H\left[x^{\left(16\rho+\frac{1}{2}\right)}G^{h,k}_{p,q}\left(\frac{\delta x^{16}}{2^{16}}\,\middle|\,\begin{matrix}a_1,\cdots\cdots,a_p\\ \beta_1,\cdots\cdots,\beta_q\end{matrix}\right)\right]^{8\gamma}=\frac{2^{(64\rho+4)}}{y^{\left(16\rho+\frac{3}{2}\right)}}G^{h,k+8}_{p+16,q}\Bigg($$

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* $\left(-\rho-\frac{\gamma}{2}+\frac{1,2,3}{4}\right)$ denotes $\left(-\rho-\frac{\gamma}{2}+\frac{1}{4}\right)$, $\left(-\rho-\frac{\gamma}{2}+\frac{1}{2}\right)$, $\left(-\rho-\frac{\gamma}{2}+\frac{3}{4}\right)$.

$$\frac{2^{48}\delta}{y^{16}}\left|\begin{matrix}-\rho-\frac{\gamma}{2}, -\rho-\frac{\gamma}{2}+\frac{1,2,3,\cdots,7}{8}, a_1,\cdots,a_p, -\rho+\frac{\gamma}{2}, -\rho+\frac{\gamma}{2}+\frac{1,2,\cdots,7}{8} \\ \beta_1, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \beta_q\end{matrix}\right), \quad (3.3)$$

$(p+q)<2(h+k)$, Re $(a_j+\rho)<\frac{1}{4}$, $j=1,\cdots\cdots k$, Re $\left(\beta_j+\rho+\frac{\gamma}{2}\right)>-\frac{1}{8}$, $j=1,\cdots,h$, and $|\arg\delta|<\left(h+k-\frac{p}{2}-\frac{q}{2}\right)\pi$.

Proceeding successively we arrive at the result

$$H\left[x^{\left(2N\rho+\frac{1}{2}\right)}G^{h,k}_{p,q}\left(\frac{\delta x^{2N}}{2^{2N}}\left|\begin{matrix}a_1,\cdots\cdots,a_p\\ \beta_1,\cdots\cdots,\beta_q\end{matrix}\right.\right)\right]^{N\gamma}=\frac{(2N)^{(2N\rho+1)}}{y^{\left(2N\rho+\frac{3}{2}\right)}}G^{h,k+N}_{p+2N,q}\left(\frac{\delta N^{2N}}{2^{2N}}\right.$$

$$\left|\begin{matrix}-\rho-\frac{\gamma}{2}, -\rho-\frac{\gamma}{2}+\frac{1,2,\cdots,(N-1)}{N}, a_1,\cdots,a_p, -\rho+\frac{\gamma}{2}, -\rho+\frac{\gamma}{2}\\ \beta_1,\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\beta_q\end{matrix}\right.$$

$$\left.+\frac{1,\ 2,\cdots\cdots,(N-1)}{N}\right), \quad (3.4)$$

where $N=2^n$, Re $n>1$, n is an integer,

$(p+q)<2(h+k)$, Re $(a_j+\rho)<\frac{1}{4}$, $j=1,\ 2,\cdots\cdots,k$, Re $\left(\beta_j+\rho+\frac{\gamma}{2}\right)>-\frac{1}{2^{n-1}}$, $j=1,\cdots,h$, and $|\arg\delta|<\left(h+k-\frac{p}{2}-\frac{q}{2}\right)\pi$.

(3.4) can be proved by mathematical induction as the theorem.

Particular Case: Let $p=k=0$, $h=q=2$, $\beta_1=\gamma$, $\beta_2=2\gamma$.

We have from (3.4)

$$H\left[x^{\left(2N\rho+3N\gamma+\frac{1}{2}\right)}K_\gamma\left(\delta^{\frac{1}{2}}\frac{x^N}{2^{N-1}}\right)\right]^{N\gamma}=\frac{2^{(2N\rho+3N\gamma)}\ N^{(2N\rho+1)}}{\delta^{\frac{3\gamma}{2}}\ y^{\left(2N\rho+\frac{3}{2}\right)}}$$

$$\times G^{2,N}_{2N,2}\left(\frac{\delta N^{2N}}{y^{2N}}\left|\begin{matrix}-\rho-\frac{\gamma}{2}, & -\rho-\frac{\gamma}{2}+\frac{1,\ 2,\cdots(N-1)}{N}, & -\rho+\frac{\gamma}{2},\\ & \gamma,\ 2\gamma. & \end{matrix}\right.\right.$$

$$\left.-\rho+\frac{\gamma}{2}+\frac{1,\ 2,\cdots(N-1)}{N}\right),$$

where $N=2^n$, Re $n>1$, n is an integer.

Re $\rho<\left(\frac{1}{4}\right)$, Re $\left(\rho+\frac{\gamma}{2}\right)>-\frac{1}{2^{n-1}}$, and $|\arg\delta|<\pi$.

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