RECURRENCE RELATIONS FOR THE GENERALIZED HANKEL TRANSFORM

By R. S. Dahiya

1. Introduction

The function $\tilde{\omega}_{\mu,\ \nu}(x)$ is defined by the integral relation

$$\tilde{\omega}_{\mu,\ \nu}(x)=x^{\frac{1}{2}}\int_0^\infty J_\mu(t)\,J_\nu\left(\frac{x}{t}\right)t^{-1}\,dt$$

$$=\frac{x^{\frac{1}{2}+\nu}2^{-2\nu-1}\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu\right)}{\Gamma(\nu+1)\,\Gamma\left(1+\frac{1}{2}\mu+\frac{1}{2}\nu\right)}\cdot F_3\left(\nu+1,1-\frac{1}{2}\mu+\frac{1}{2}\nu,\frac{1}{2}\mu+\frac{1}{2}\nu+1;\frac{x^{-}}{16}\right)$$

+*another term with* μ *and* ν *interchanged*;

$$-R\left(\mu+\frac{3}{2}\right)<0<R\left(\nu+\frac{3}{2}\right)$$

The above function has been generalized to n parameters as :

$$\tilde{\omega}_{\mu_1,\cdots\cdots,\ \mu_n}(x)=x^{\frac{1}{2}}\int_0^\infty\cdots\cdots\int_0^\infty J_{\mu_1}(t_1)\cdots\cdots J_{\mu_{n-1}}(t_{n-1})\,J_{\mu_n}\left(\frac{x}{t_1\cdots\cdots t_{n-1}}\right)$$

$$\cdot(t_1\cdots\cdots t_{n-1})^{-1}\,dt_1\cdots\cdots dt_{n-1}$$

$$=\int_0^\infty\tilde{\omega}_{\mu_1,\ \cdots\cdots,\ \mu_{n-1}}\left(\frac{x}{t}\right)J_{\mu_n}(t)^{-\frac{1}{2}}\,dt,$$

where $R(\mu_k+2)\geq 0$, $k=1,\ 2,\ \cdots\cdots,\ n$ and the μ's may be permuted among themselves.

V.P. Mainva [2] in 1958 defined the function

$$\tilde{\omega}_{\mu,\nu}\ \lambda(x)=\int_0^\infty\tilde{\omega}_{\mu,\nu}(xt)\ J_\lambda(t)\ \sqrt{t}\ dt$$

$$=\frac{2^{-\mu}\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(1+\frac{1}{2}\lambda+\frac{1}{2}\mu\right)x^{\mu+\frac{1}{2}}}{\Gamma(1+\mu)\,\Gamma\left(1+\frac{1}{2}\nu+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\lambda-\frac{1}{2}\mu\right)}$$

$$\cdot 2F3\left[\begin{matrix}1+\frac{\lambda}{2}+\frac{\mu}{2},1-\frac{\lambda}{2}+\frac{\mu}{2}: & \\ 1+\mu,1\pm\frac{\nu}{2}+\frac{\mu}{2}: & -\frac{x^2}{4}\end{matrix}\right]$$

+*a similar expression with* μ *and* ν *interchanged*;

and further generalized this kernel as

$$\tilde{\omega}^{\nu_1,\cdots\cdots,\nu_m}_{\mu_1,\cdots\cdots,\mu_n}(x)=\int_0^\infty\cdots\cdots\int_0^\infty J_{\nu_1}(t_1)\cdots\cdots J_{\nu_m}(t_m)\ J_{\mu_1}(T_1)\cdots\cdots J_{\mu_{n-2}}(T_{n-2})$$

$$\cdot\tilde{\omega}_{\mu_{n-1},\mu_n}\left(\frac{xt_1\cdots\cdots t_m}{T_1\cdots\cdots T_{n-2}}\right)\left(\frac{(t_1\cdots\cdots t_m)^{\frac{1}{2}}}{T_1\cdots\cdots T_{n-2}}\right)dT_1\cdots dT_{n-2}\ dt_1\cdots dt_m,$$

$$m<n,\quad R(\mu_r,\ \nu_s)>-1,\quad r=1,\ 2,\ \cdots\cdots,\ n.$$

In this paper I shall prove certain recurrence relations for the above functions. These relations are the generalization of the previous results.

2. Results to be used in this paper [3 ; 2]

$$(2\cdot1)\quad \int_0^\infty \tilde{\omega}_{\mu_1,\cdots\cdots,\mu_r}(xt)\ \tilde{\omega}_{\mu_{r+1},\cdots\cdots\mu_n}(yt^{-1})\ \frac{dt}{t}=\tilde{\omega}_{\mu_1,\cdots\cdots\mu_n}(xy)$$

$$R(\mu_r)>-1,\quad r=1,\ 2,\ \cdots\cdots,\ n.$$

$$(2\cdot2)\quad \int_0^\infty x^{-1}\ J_{2\nu-1}\left(\frac{2}{\sqrt{x}}\right)\tilde{\omega}_{\mu_1,\cdots\cdots,\mu_n}(ax)\ dx=\tilde{\omega}_{\mu_1,\cdots\cdots,\mu_n,\ \nu,\ \nu-1}(a)$$

$$R(\mu_r)>-\frac{3}{4},\quad r=1,\ 2,\ \cdots\cdots,\ n,\quad R(\nu)>\frac{1}{2n}.$$

$$(2\cdot3)\quad \int_0^\infty x^{-1}\ J_{2\nu-1}\left(\frac{2}{\sqrt{x}}\right)\tilde{\omega}^{\nu_1,\cdots\cdots,\nu_m}_{\mu_1,\cdots\cdots,\mu_n}(ax)\ dx=\tilde{\omega}^{\nu_1,\cdots\cdots,\nu_m}_{\mu_1,\cdots\cdots,\mu_n,\ \nu,\ \nu-1}(a),$$

$$m<n,\quad R(\mu_r)>-\frac{3}{4},\quad r=1,\ 2,\cdots\cdots,n,\quad R(\nu)>\frac{1}{2(n-m)}.$$

$$(2\cdot4)\quad \int_0^\infty \tilde{\omega}_{\mu_1,\cdots\cdots,\mu_n}(xt)\ \tilde{\omega}_{\nu_1,\cdots\cdots,\nu_m}(t)dt=\tilde{\omega}^{\nu_1,\cdots\cdots,\nu_m}_{\mu_1,\cdots\cdots,\mu_n}(x),\qquad m<n.$$

$$(2\cdot5)\quad \int_0^\infty x^{-m-\frac{3}{2}}\ H_{-m-\frac{1}{2}}\left(\frac{a}{x}\right)\tilde{\omega}_{\mu_1,\cdots\cdots,\mu_n}(px)dx=\sqrt{a}\ \frac{d^m}{da^m}\left[a^{-1}\ \tilde{\omega}_{\mu_1,\cdots\cdots,\mu_n,\ 1/2}(a^p)\right],$$

$$R(\mu_r)>-1,\quad R(\mu_r+m)>-\frac{3}{2},\quad r=1,\ 2,\ \cdots\cdots,\ n.$$

$$(2\cdot6)\quad \int_0^\infty x^{-m-\frac{3}{2}}\ H_{-m-\frac{3}{2}}\left(\frac{a}{x}\right)\tilde{\omega}^{\nu_1,\cdots\cdots,\nu_r}_{\mu_1,\cdots\cdots,\mu_n}(px)dx=\sqrt{a}\frac{d^m}{da^m}\left[a^{-1}\ \tilde{\omega}^{\nu_1,\cdots\cdots,\nu_m}_{\mu_1,\cdots\cdots,\mu_n,1/2}(ap)\right]$$

$$n>r,\quad R(\mu_s)>-1,\quad R(\mu_s+m)>-\frac{3}{2},\quad s=1,\ 2,\ \cdots\cdots,\ n.$$

$$(2\cdot7)\quad (\mu-\nu)\left[\tilde{\omega}_{\mu-1,\nu-1}(Z)-\tilde{\omega}_{\mu+1,\nu+1}(Z)\right]+(\mu+\nu)$$

$$\cdot\left[\tilde{\omega}_{\mu-1,\nu+1}(Z)-\tilde{\omega}_{\mu+1,\nu-1}(Z)\right]=0$$

(2·8) $$(\mu-\nu)\Big[\tilde{\omega}^{\lambda}_{\mu-1,\nu-1}(Z)-\tilde{\omega}^{\lambda}_{\mu+1,\nu+1}(Z)\Big]+(\mu+\nu)\cdot\Big[\tilde{\omega}^{\lambda}_{\mu-1,\nu+1}(Z)-\tilde{\omega}^{\lambda}_{\mu+1,\nu-1}(Z)\Big]=0$$

(2·9) $$(\nu+\lambda)\Big[\tilde{\omega}^{\lambda}_{\mu,\nu-1}(Z)-\tilde{\omega}^{\lambda+1}_{\mu,\nu+1}(Z)\Big]+(\lambda-\nu)\Big[\tilde{\omega}^{\lambda-1}_{\mu,\nu+1}(Z)-\tilde{\omega}^{\lambda+1}_{\mu,\nu-1}(Z)\Big]=0$$

(2·10) $$2J_\nu(Z)-J_{\nu-1}(Z)+J_{\nu+1}(Z)=0$$

3. Recurrence relations to be proved.

(3·1) $$(\mu-\nu)\Big[\tilde{\omega}_{\mu-1,\,\nu-1,\nu_1,\,\ldots\ldots,\,\nu_s}(Z)-\tilde{\omega}_{\mu+1,\,\nu+1,\nu_1,\,\ldots\ldots,\,\nu_s}(Z)\Big]+(\mu-\nu)\Big[\tilde{\omega}_{\mu-1,\nu+1,\nu_1,\,\ldots\ldots,\,\nu_s}(Z)-\tilde{\omega}_{\mu+1,\nu-1,\nu_1,\,\ldots\ldots,\,\nu_s}(Z)\Big]=0$$

(3·2) $$2\tilde{\omega}_{\mu_1,\,\ldots\ldots,\,\mu_n,\,\nu-\frac{1}{2},\,\nu-\frac{3}{2}}(Z)-\tilde{\omega}_{\mu_1,\,\ldots\ldots,\,\mu_n,\,\nu,\,\nu-1}(Z)+\tilde{\omega}_{\mu_1,\,\ldots\ldots,\,\mu_n,\,\nu+\frac{1}{2},\,\nu-\frac{1}{2}}(Z)=0$$

(3·3) $$2\tilde{\omega}^{\nu_1,\,\ldots\ldots,\,\nu_m}_{\mu_1,\,\ldots\ldots,\,\mu_n,\,\nu-\frac{1}{2},\,\nu-\frac{3}{2}}(Z)-\tilde{\omega}^{\nu_1,\,\ldots\ldots,\,\nu_m}_{\mu_1,\,\ldots\ldots,\,\mu_n,\,\nu,\,\nu-1}(Z)+\tilde{\omega}^{\nu_1,\,\ldots\ldots,\,\nu_m}_{\mu_1,\,\ldots\ldots,\,\mu_n,\,\nu+\frac{1}{2},\,\nu-\frac{1}{2}}(Z)=0$$

(3·4) $$(\mu-\nu)\Big[\tilde{\omega}^{\mu-1,\nu-1}_{\nu_1,\,\ldots\ldots,\,\nu_s}(Z)-\tilde{\omega}^{\mu+1,\nu+1}_{\nu_1,\,\ldots\ldots,\,\nu_s}(Z)\Big]+(\mu+\nu)\Big[\tilde{\omega}^{\mu-1,\nu+1}_{\nu_1,\,\ldots\ldots,\,\nu_s}(Z)-\tilde{\omega}^{\mu+1,\nu-1}_{\nu_1,\,\ldots\ldots,\,\nu_s}(Z)\Big]=0$$

(3·5) $$(\mu-\nu)\,[\tilde{\omega}_{\mu-1,\nu-1,\nu_1,\nu_1-1}(Z)-\tilde{\omega}_{\mu+1,\nu+1,\nu_1,\nu_1-1}(Z)]+(\mu+\nu)\,[\tilde{\omega}_{\mu-1,\nu+1,\nu_1,\nu_1-1}(Z)-\tilde{\omega}_{\mu+1,\nu-1,\nu_1,\nu_1-1}(Z)]=0$$

(3·6) $$(\mu-\nu)\Big[\tilde{\omega}^{\lambda}_{\mu-1,\nu-1,\nu_1,\nu_1-1}(Z)-\tilde{\omega}^{\lambda}_{\mu+1,\nu+1,\nu_1,\nu_1-1}(Z)\Big]+(\mu+\nu)\Big[\tilde{\omega}^{\lambda}_{\mu-1,\nu-1,\nu_1,\nu_1-1}(Z)-\tilde{\omega}^{\lambda}_{\mu+1,\nu+1,\nu_1,\nu_1-1}(Z)\Big]=0$$

(3·7) $$(\nu+\lambda)\Big[\tilde{\omega}^{\lambda-1}_{\mu,\nu-1,\nu_1,\nu_1-1}(Z)-\tilde{\omega}^{\lambda+1}_{\mu,\,\nu+1,\nu_1,\nu_1-1}(Z)\Big]+(\lambda-\nu)\Big[\tilde{\omega}^{\lambda-1}_{\mu,\nu+1,\nu_1,\nu_1-1}(Z)-\tilde{\omega}^{\lambda+1}_{\mu,\nu-1,\nu_1-1,\nu_1}(Z)\Big]=0$$

(3·8) $$(\mu-\nu)\frac{d^m}{dZ^m}\Big[\frac{1}{2}\tilde{\omega}_{\mu-1,\nu-1,\frac{1}{2}}(Z)-\frac{1}{2}\tilde{\omega}_{\mu+1,\nu+1,\frac{1}{2}}(Z)\Big]+(\mu+\nu)\frac{d^m}{dZ^m}\Big[\frac{1}{2}\tilde{\omega}_{\mu-1,\nu+1,\frac{1}{2}}(Z)+\frac{1}{2}\tilde{\omega}_{\mu+1,\nu-1,\frac{1}{2}}(Z)\Big]=0$$

(3·9) $$(\nu+\lambda)\frac{d^m}{dZ^m}\Big[\frac{1}{2}\tilde{\omega}_{\mu,\nu-1,\frac{1}{2}}(Z)-\frac{1}{2}\tilde{\omega}^{\lambda+1}_{\mu,\nu+1,\frac{1}{2}}(Z)\Big]+(\lambda-\nu)\frac{d^m}{dZ^m}\Big[\frac{1}{2}\tilde{\omega}^{\lambda-1}_{\mu,\nu+1,\frac{1}{2}}(Z)-\frac{1}{2}\tilde{\omega}^{\lambda+1}_{\mu,\nu-1,\frac{1}{2}}(Z)\Big]=0$$

(3·10) $$(\mu-\nu)\frac{d^m}{dZ^m}\Big[\frac{1}{2}\tilde{\omega}^{\lambda}_{\mu-1,\nu-1,\frac{1}{2}}(Z)-\frac{1}{2}\tilde{\omega}^{\lambda}_{\mu+1,\nu+1,\frac{1}{2}}(Z)\Big]$$

$$+(\mu+\nu)\frac{d^m}{dZ^m}\left[\frac{1}{2}\tilde{\omega}^{\lambda}_{\mu-1,\nu+1,\frac{1}{2}}(Z)-\frac{1}{2}\tilde{\omega}^{\lambda}_{\mu+1,\nu-1,\frac{1}{2}}(Z)\right]=0$$

PROOFS. The relation (3·1) can be proved by making use of (2·1) and (2·5). Multiply (2·5) by $\tilde{\omega}_{\nu_1,\dots\dots,\nu_n}(yz)$ and integrating between the limits $(0, \infty)$ throughout the equation. Now use (2·1) to get the desired result. Similarly other recurrence relations can be proved by making use of the remaining results.

Iowa State University
U. S. A.

REFERENCES

[1] G. N. Watson; *A treatise on the theory of Bessel Functions*, Camb. Univ. Press(1966).

[2] V. P. Mainva; *A New transform*, Bull. Cal. Math. Soc., 1958 supplement 76—93.

[3] S.C. Mitra; *On a theorem in the greneralized Fourier transform*, Carad. Math. Bull. Vol. 10. No. 5, 1967.

[4] K.P. Bhatnager; (i) *On certain theorems on self-reciprocal functions*, Acad. Roy. Belgique Bull. cl. Sci. (5) 39(1953) 42—69. (ii) *Two theorems on self-reciprocal functions onic a new transform*. Bull. Cal. Math. Soc. 45, (1953) 109—112.