ON SYMMETRIC GENERALIZED UNIFORM SPACES

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In this paper we provide answers to four questions raised by the author in [1].

(Q_1) Does every symmetric generalized uniform space have an open base? No.

(Q_2) Does there exist a symmetric generalized uniform space which has a convergent filter which is not Cauchy? Yes.

(Q_3) Is there a proximity class of symmetric generalized uniformities with more than two totally bounded elements? Yes.

(Q_4) Does there exist a totally bounded uniformity which is not p-correct or p-correct of degree n? Yes.

Let R denote the reals, let I denote the positive integers. Let $\Delta=\{(x,y) \mid x=y \in R\}$. Let $\Delta^*=\{(x,y) \mid x=-y \in R\}$. Let $I_n=[-1/n, 1/n]$ for each $n \in I$. Let $B_n=((I_n \times I_n)-\Delta^*) \cup \Delta$ for each $n \in I$. Let $\mathscr{B}=\{B_n \mid n \in I\}$.

THEOREM. *$\mathscr{B}$ is a base for a symmetric generalized uniformity $\mathscr{U}$ on R that has the following properties*:

(1) *For every U, V in $\mathscr{U}$ $(U \cap V) \in \mathscr{U}$.*

(2) *$(0,0) \notin B_n^0$ for every $n \in I$; so that $\mathscr{U}$ does not have an open base, and $\mathscr{U}$ is not p-correct.*

(3) *$(B_n \circ B_n) \cap ((R \times R)-B_m) \neq \phi$ for every m, n in I.*

(4) *The neighborhood system of 0 is a convergent filter in $(R, \mathscr{T}(\mathscr{U}))$ that is not Cauchy with respect to $\mathscr{U}$.*

(5) *$(R, \mathscr{U})$ is complete.*

(6) *$\mathscr{T}(\mathscr{U})$ is not compact; so that $\mathscr{U}$ is not totally bounded.*

PROOF. The proof that $\mathscr{B}$ is a base for a symmetric generalized uniformity on R is straightfoward, for suppose $b \in B_n[A] \cup B$. If $b \neq 0$, then there exists $m \in I$ such that $B_m[b]=b$ (choose m such that $1/m<|b|$). If $b=0$ then there exists $a_n \in A \cap [-1/n, 1/n]$ such that $(a_n, 0) \in B_n$. If $a_n=0$, then $B_n[0] \subset B_n[A]$. If $a_n \neq 0$, then for any $m \in I$ such that $1/m<|a_n|$ we have that $B_m[0] \subset B_n[A]$.

PROOF 1. $(B_n \cap B_m) = B_m$ if $m \geq n$.

PROOF 2. $(0_1 \times 0_2) \cap (\Delta^* - (0,0)) \neq \phi$ for every $0_1, 0_2$ in $\mathscr{N}(0)$, the neighborhood system of 0.

PROOF 3. $(B_n \circ B_n) \cap (\Delta^* - (0,0)) \neq \phi$ for every $n \in I$.

PROOF 4. Same as proof of 2.

PROOF 5. Let $\mathscr{F}$ be a weakly Cauchy filter in $(R, \mathscr{T}(\mathscr{U}))$.
For every $n \in I$ there exists $x_n \in R$ such that $B_n(x_n) \in \mathscr{F}$. Suppose for some $m \in I$ $x_m \in (X - [-1/m, 1/m])$. Then $\mathscr{F} \supset \mathscr{N}(x_m)$, the neighborhood system of x_m. Suppose for every $n \in I$ we have that $-1/n \leq x_n \leq 1/n$. Then 0 is a cluster point for $\mathscr{F}$.

PROOF 6. $\{x\}$ is open if $x \neq 0$, and for each $n \in I$ $(-1/n, 1/n)$ is an open neighborhood of 0.

By a suitable modification of the above construction it is possible to prove the following

THEOREM. *There exists a symmetric generalized uniformity $\mathscr{U}$ on R without an open base that generates the usual topology for R such that for every U, V in $\mathscr{U}$ $(U \cap V) \in \mathscr{U}$.*

THEOREM. *There exists a totally bounded symmetric generalized uniform space that is not p-correct or p-correct of degree n for every $n \in I$.*

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REFERENCES

[1] C. J. Mozzochi, *Symmetric generalized uniform and proximity spaces*, a publication of the Department of Mathematics, Trinity College, Hartford, Connecticut, October 1968.