CERTAIN THEOREMS ON GENERALIZED HANKEL TRANSFORM

By R. S. Dahiya

1. In this paper I have discussed a property of a generalization of Hankel transform whose recurrence relations, integral representations and other properties have been given by me and by some other auothers. This property involves its connection with the Laplace, Hankel and K-transforms. I have used this property to obtain certain theorems and to evaluate certain definite integrals from there, which are difficult to tackel otherwise.

2. Agarwal (1950) gave a generalization of the well-known Hankel transform, viz.,

$$f(x)=\int_0^\infty \sqrt{xy}\, J_v(xy)\; g(y)\; dy, \tag{2·1}$$

by means of the integral equations

$$f(x)=(1/2)^{\lambda}\int_0^\infty (xy)^{\lambda+\frac{1}{2}} J_\lambda^\mu\left(\frac{1}{4}x^2y^2\right) g(y)dy, \tag{2·2}$$

where $J_\lambda^\mu(x)$ is the Bessel-Maitland function defined by the series

$$J_\lambda^\mu(x)=\Sigma_{\gamma=0}^{\infty}\frac{(-x)^\gamma}{\gamma!\Gamma(1+\lambda+\mu\gamma)},\quad \mu>0. \tag{2·3}$$

For $\mu=1$, (2·2) reduces to (2·1).

He gave the inversion formula (1950) and some theorems (1951) for this generalized Hankel-transform.

we can write (2·2) in the convenient form

$$f(x)=\int_0^\infty (xy)^\gamma J_\lambda^\mu(xy)\; g(y)\; dy. \tag{2·4}$$

The object of this paper is to evaluate some general integrals involving Bessel-Maitland function defined by (2·3). Henceforth we shall call $f(x)$ to be generalized Hankel-transform of $g(y)$, if the relation (2·4) holds.

3. THEOREM 1. *If $f(x)$ is generalized Hankel-transform of $g(y)$ and $\frac{1}{\mu}y^{-(\lambda+1)+\mu(\gamma+1)}g(y^\mu)$ is the Laplace-transform of $t^{(\alpha+1)\mu-1}\Psi(t^{\mu/2})$ and*

$t^q\Psi(1/t)$ is the Hankel-transform of order v of $s^p\,\phi(s)$, then

$$f(x)=\frac{2\mu\Gamma\left(\frac{\lambda+1}{\mu}-\alpha-\frac{q}{2}+\frac{v}{2}+\frac{1}{8}\right)}{\Gamma(v+1)\;x^{\frac{\lambda+1}{\mu}-\alpha-q/2-\gamma-3/8}}\int_0^\infty s^{p-\frac{1}{2}}e^{-\frac{s^2}{8x}}\,M_{\frac{\lambda+1}{\mu}-\alpha-\frac{q}{2}-\frac{3}{8},\ \frac{v}{2}}\left(\frac{s^2}{4x}\right)\times\phi(s)\;ds \tag{3·1}$$

provided $R\left(\frac{\lambda+1}{\mu}-\frac{q}{2}-\alpha+\frac{v}{2}\right)>-\frac{1}{8}$ *and the integrals involed converge absolutely.*

In particular, if $q=2\left[\frac{\lambda+1}{2}-\alpha-\frac{v}{2}-\frac{7}{8}\right]$, *we obtain*

$$x^{\gamma-v}f\left(\frac{1}{4x}\right)\doteqdot\frac{\mu}{2^{2\gamma-v-1}}\;s^{\frac{v+p}{2}-\frac{1}{4}}\,\phi(\sqrt{s}) \tag{3·2}$$

PROOF. For, if $\frac{1}{\mu}\;y^{-(\lambda+1)+\mu(\gamma+1)}g(y^{\mu})$ is the Laplace-transform of $t^{(\alpha+1)\mu-1}\Psi(t^{\mu/2})$, then

$$y^{-\frac{\lambda+1}{\mu}+\gamma+1}g(y)=\int_0^\infty t^{\alpha}e^{-(yt)^{\frac{1}{\mu}}}\Psi(\sqrt{t})\;dt$$

and

$$f(x)=\int_0^\infty(xy)^{\gamma}J_{\lambda}^{\mu}(xy)g(y)dy$$

$$=\int_0^\infty(xy)^{\gamma}J_{\lambda}^{\mu}(xy)y^{\frac{\lambda+1}{\mu}-\gamma-1}\int_0^\infty t^{\alpha}e^{-(yt)^{\frac{1}{\mu}}}\Psi(\sqrt{t})dt\;dy,$$

$$=x^{\gamma}\int_0^\infty t^{\alpha}\Psi(\sqrt{t})\int_0^\infty y^{\frac{\lambda+1}{\mu}-1}e^{-(yt)^{\frac{1}{\mu}}}J_{\lambda}^{\mu}(xy)\;dy\;dt, \tag{3.3}$$

Provided we can justify the change in the order of integration,

$$=\mu\;x^{\gamma}\int_0^\infty t^{\frac{\lambda+1}{\mu}-\alpha-2}e^{-xt}\Psi\left(\frac{1}{\sqrt{t}}\right)\;dt.$$

on expending $J_{\lambda}^{\mu}(xy)$ in the form of infinite series and integrating term by term, a process easily justifiable,

$$f(x)=\mu x^{\gamma}\int_0^\infty s^{p+\frac{1}{2}}\phi(s)\int_0^\infty t^{\frac{\lambda+1}{\mu}-\alpha-q/2-7/8}J_v(\sqrt{t}\,s)\;dt\;ds, \tag{3·4}$$

Provided we can justify the change in the order of integration;

$$f(x)=\frac{2\mu\Gamma\left(\frac{\lambda+1}{\mu}-\alpha-\frac{q}{2}+\frac{v}{2}+\frac{1}{8}\right)}{\Gamma(v+1)x^{\frac{\lambda+1}{\mu}-\alpha-\frac{q}{2}-\gamma-\frac{3}{8}}}\int_0^\infty s^{p-\frac{1}{2}}e^{-\frac{s^2}{8x}}$$

$$\times M_{\frac{\lambda+1}{\mu}-\alpha-\frac{q}{2}-\frac{3}{8},\ \frac{v}{2}}\left(\frac{s^2}{4x}\right)\phi(s)ds.$$

since (i) as $x\to\infty$,

$$J_\lambda^\mu(x)=o\left[x^{-k(\lambda+\frac{1}{2})}\exp\{(\mu x)^k\cos\pi k/\mu k\}\right]_{k=\frac{1}{1+\mu}},$$

(ii) as $x\to 0$, $J_\lambda^\mu(x)=o(1)$, the change in the order of integration in the steps (3·2) and (3·3) is permissible under the conditions stated by virtue of De La Vallee Poussin's theorem.

4. Applications. (a) Suppose $\mu=1$ and $g(y^2)=y^{2\lambda-2\gamma-2\beta-1}e^{y^2/4}W_{k,\beta}\left(\frac{y^2}{2}\right)$

$$\therefore\ f(x)=\frac{\Gamma(1+\lambda-2\beta)}{\Gamma(1/2+\beta-k)}2^{\frac{\lambda-\beta-k}{2}-\frac{1}{4}}x^{\frac{\gamma+\beta-k}{2}-\frac{3}{4}}e^xW_{\frac{k+3\beta-\lambda}{2}-\frac{1}{4},\ \frac{k-\beta+\lambda}{2}+\frac{1}{4}}(2x)$$

and $t^q\Psi\left(\frac{1}{t}\right)=\dfrac{t^{q-4\beta+2\alpha+2}(t^2+2)^{\beta+k-\frac{1}{2}}}{2^k\Gamma\left(\frac{1}{2}-k+\beta\right)}$

If we take $k=-2\beta-v-\frac{1}{2}$ and $q=4\beta-2\alpha+v-3/2$, then

$$s^p\phi(s)=\frac{s^{\beta+v+\frac{1}{2}}K_\beta(2\sqrt{s})}{2^{v+3/2\beta}\Gamma(\beta+v+1)}$$

Hence from the theorem, we get

$$\int_0^\infty s^{\beta+v}e^{-s^2/8x}M_{\lambda-2\beta-\frac{v}{2}+\frac{11}{8},\ \frac{v}{2}}\left(\frac{s^2}{4x}\right)K_\beta(\sqrt{2}s)ds$$

$$=\frac{2^{\frac{3v}{2}+\frac{\lambda}{2}+2\beta-1}\Gamma(v+1)\Gamma(1+\lambda-2\beta)}{\Gamma(\lambda-2\beta+\frac{15}{8})\Gamma(1+v+3\beta)}x^{\frac{\lambda-\beta}{2}+\frac{7}{8}}e^xW_{\frac{\beta-\lambda-v-1}{2},\ \frac{\lambda-3\beta-v}{2}}.$$

(b) suppose $\mu=1$ and $g(y^2)=y^{\lambda-2v-1}\exp(-\sqrt{2}y)$.

$$\therefore\ f(x)=2^{\frac{1-\lambda}{2}}x^{\gamma-\lambda}(1+2x)^{-1/2}[\sqrt{1+2x}-1]^\lambda$$

and $t^q\Psi\left(\frac{1}{t}\right)=t^{q+2\alpha-\lambda+1}\exp\left(-\frac{t^2}{4}\right)D_{-\lambda}(t^2)$.

suppose if we take $q=\lambda-2\alpha+v-\frac{1}{2}$, then

$$s^{p}\phi(s)=\frac{2^{\frac{\lambda}{2}}}{\sqrt{\pi}}s^{-\frac{\lambda}{2}-1}e^{-\frac{s^2}{4}}M_{\frac{v}{2}-\frac{\lambda}{4}+\frac{3}{4},\ \frac{v}{2}+\frac{\lambda}{4}+\frac{1}{4}}\left(\frac{s^2}{4}\right).$$

Hence from the theorem, we get

$$\int_0^\infty s^{-\frac{\lambda}{2}-\frac{3}{2}}\exp\left[\left(-\frac{s^2}{8x}\right)-\frac{s^2}{4}\right]M_{\frac{\lambda-v}{2}+\frac{7}{8},\ \frac{v}{2}}\left(\frac{s^2}{4x}\right)M_{\frac{v}{2}-\frac{\lambda}{4}+\frac{3}{4},\ \frac{v}{2}+\frac{\lambda}{4}+\frac{1}{4}}\left(\frac{s^2}{2}\right)ds$$

$$=\frac{\sqrt{\pi}\Gamma(1+v)x^{\frac{v}{2}+\frac{3\lambda}{4}+\frac{3}{4}}}{2^{\frac{\lambda+1}{2}}\Gamma\left(v-\frac{\lambda}{4}+\frac{5}{4}\right)}\cdot\frac{[\sqrt{1+2x}-1]^{\lambda}}{\sqrt{1+2x}}\,;\ R(v)>-1,\ R\left(v-\frac{\lambda}{4}\right)>-\frac{5}{4}.$$

(c) Suppose $\mu=1$ and $g(y^2)=y^{2\lambda-2\gamma-1}e^{y^2/\mu}D_{-2\lambda}(y)$

$$\therefore\ f(x)=2^{\lambda}x^{\gamma-\frac{1}{2}}e^{x}D_{-2\lambda}(2\sqrt{x})$$

and $$t^{q}\psi\left(\frac{1}{t}\right)=\frac{t^{q+2\alpha+1}}{\Gamma\left(\lambda+\frac{1}{2}\right)(2+t^2)^{\lambda}}.$$

Suppose if we take $\lambda=1$ and $q=v-2\alpha-\frac{3}{2}$, then

$$s^{p}\phi(s)=2^{\frac{v-1}{2}}\sqrt{\pi}\ \sec(v\pi)\sqrt{s}\,[I_v(\sqrt{2}s)-L_{-v}(\sqrt{2}s)].$$

Hence from the theorem, we get

$$\int_0^\infty\exp\left(-\frac{s^2}{8x}\right)M_{\frac{19}{8}-\frac{v}{2},\ \frac{v}{2}}\left(\frac{s^2}{4x}\right)[I_v(\sqrt{2}s)-L_{-v}(\sqrt{2}s)]\,ds$$

$$=\frac{\Gamma(v+1)e^{x}x^{\frac{15}{8}-\frac{v}{2}}D_{-2}(2\sqrt{x})}{\sqrt{\pi}\Gamma\left(\frac{23}{8}\right)\sec(v\pi)2^{(v-1)/2}}$$

In particular, if we take $\lambda=\beta+v+1$ and $q=v-2\alpha-\frac{1}{2}$, then we obtain

$$\int_0^\infty s^{\beta+v}\exp\left(-\frac{s^2}{8x}\right)M_{\beta+\frac{v}{2}+\frac{15}{8},\ \frac{v}{2}}\left(\frac{s^2}{4x}\right)K_{\beta}(\sqrt{2}s)\,ds$$

$$=\frac{\Gamma(v+1)\Gamma\left(\beta+v+\frac{3}{2}\right)\Gamma(\beta+v+1)2^{2v+\frac{s\beta}{2}}}{\Gamma\left(\beta+v+\frac{19}{8}\right)}e^{x}x^{\beta+\frac{v}{2}+\frac{11}{8}}D_{-2(\beta+v+1)}(2\sqrt{x}).$$

5. THEOREM 2. *If $f(x)$ is the generalized Hankel-transform of $g(y)$, and $y^{\lambda}g(y)$ is the K-transform of $t^{\alpha\mu-\frac{3}{2}}\phi(t^{\mu})$ of order v, then*

$$f(x)=2x^{\gamma}\int_0^{\infty}t^{\alpha-1}\phi(t)\left\{\Sigma\frac{2^{\gamma-\lambda-\frac{1}{2}+r}(-x)^r\Gamma\left(\frac{\gamma-\lambda\pm v+r}{2}+\frac{3}{4}\right)}{t^{\frac{\gamma-\lambda+r}{\mu}+\frac{3}{2\mu}}\Gamma(r+1)\Gamma(1+\lambda+\mu r)}\right\}dt\ .$$

In particular, If we take $\mu=1/2$, we obtain

$$f(x)=2^{\gamma-\lambda+\frac{1}{2}}x^{\gamma}\int_0^{\infty}t^{2(\lambda-\gamma)+\alpha-4}\phi(t)\left\{\frac{\Gamma\left(\frac{\gamma-\lambda\pm v}{2}+\frac{3}{4}\right)}{\Gamma(1+\lambda)}{}_2F_3\left(\begin{matrix}\gamma-\lambda\pm v+\frac{3}{4}\ ; & -x^2/t^4\\ \frac{1}{2},1,1+\lambda\ ; & \end{matrix}\right)\right.$$

$$\left.+\frac{2x\Gamma\left(\frac{\gamma-\lambda\pm v}{2}+\frac{5}{4}\right)}{t^2\Gamma\left(\frac{3}{2}+\lambda\right)}{}_2F_3\left(\frac{\gamma+\lambda\pm v}{2}+\frac{3}{4}\ ;\ \frac{1}{2},\ 1,\ 1+\lambda\ ;\ -x^2/t^4\right)\right\}dt.$$

The proof of the theorem is on the same line.

6. **Applications.** (i) Let $g(y)=y^{\lambda+v-\gamma+\frac{1}{2}}K_{\lambda-v+\frac{1}{2}}(y)$

$$\therefore\ f(x)=2^{\lambda-v-\frac{1}{2}}\Gamma(2v+1)x^{\gamma-\frac{1}{2}}e^{\frac{x^2}{2}}W_{-v-\frac{1}{2},\ \frac{1}{4}}(x^2).$$

Suppose $\gamma=2\lambda+1$, then $\phi(t)=\dfrac{2^v t^{4-\alpha}}{(t^4-1)^{\frac{v}{2}}}P^v_{\lambda-v-\frac{1}{2}}(2t^4-1)$, when $1<t<\infty$

$=0,$ when $0<t<1$

Hence from the theorem, we get

$$\int_1^{\infty}t^{-2\lambda-2}(t^2-4)^{-v/2}P^v_{\lambda-v-\frac{1}{2}}(2t^4-1)\left\{\frac{\Gamma\left(\frac{\lambda\pm v}{2}+\frac{5}{4}\right)}{\Gamma(1+\lambda)}{}_2F_3\left(\begin{matrix}\frac{\lambda\pm v}{2}+\frac{5}{4}\ ; & \\ & -x^2/t^4\\ \frac{1}{2},\ 1,\ 1+\lambda\ ; & \end{matrix}\right)\right.$$

$$\left.+\frac{2x\Gamma\left(\frac{\lambda\pm v}{2}+\frac{7}{4}\right)}{t^2\Gamma\left(\frac{3}{2}+\lambda\right)}{}_2F_3\left(\begin{matrix}\frac{\lambda\pm v}{2}+\frac{7}{4}\ ; & \\ & -x^2/t^4\\ 1,\ \frac{3}{2},\ \frac{3}{2}+\lambda\ ; & \end{matrix}\right)\right\}dt$$

$$=2^{-2v-2}\Gamma(2v+1)x^{2\lambda+\frac{1}{2}}\exp(x^2/2)W_{-v-\frac{1}{2},\ \frac{1}{4}}(x^2)\ ;\ R(v)>-\frac{1}{2}.$$

(ii) Let $g(y)=\dfrac{y^{2\lambda-\gamma+1}}{1+y^2}$ and $\gamma=3\lambda+v+\frac{3}{2}$, then

$$\phi(t)=t^{4-\alpha}H_v(t^2).$$

Hence from the theorem, we get

$$\int_0^\infty t^{-4\lambda-2v-3} H_v(t^2)\left\{\frac{\Gamma\left(\frac{2\lambda+v\pm v+3}{2}\right)}{\Gamma(1+\lambda)}\, {}_2F_3\left(\begin{matrix}\frac{2\lambda+v\pm v+3}{2}; \\ \frac{1}{2},\ 1,\ 1+\lambda;\end{matrix}\quad -x^2/t^4\right)\right.$$

$$\left.+\frac{2x\Gamma\left(\frac{2\lambda+v\pm v}{2}+2\right)}{t^2\Gamma(3/2+\lambda)}\, {}_2F_3\left(\begin{matrix}\frac{2\lambda+v\pm v}{2}+2; \\ 1,\ \frac{3}{2},\ \frac{3}{2}+\lambda;\end{matrix}\quad -x^2/t^4\right)\right\} dt$$

$$=\frac{\pi^{3/2}\operatorname{cosec}(-2\lambda\pi)}{x^{\frac{11\lambda}{3}+v+\frac{1}{6}}\, 2^{\frac{4\lambda+3v+7}{3}}}\left\{J_{\lambda,-\frac{1}{2}}\left[3\left(\frac{x}{2}\right)^{2/3}\right]\cos(-\lambda\pi)\right.$$

$$\left.+J_{\lambda+\frac{1}{2},\frac{1}{2}}\left[3\left(\frac{x}{2}\right)^{2/3}\right]\sin(-\lambda\pi)-J_{-\lambda-\frac{1}{2},-\lambda}\left[3\left(\frac{x}{2}\right)^{2/3}\right]\right\}; \qquad R(\lambda)>-1.$$

Iowa State University
U. S. A.

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