ON CHAINS OF MEIJER-LAPLACE TRANSFORM OF TWO VARIABLES

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1. Introduction

The well known Laplace transform of two variables [2, p. 39]

$$(1.1)\qquad F(p,q)=pq\iint_0^{\infty(*)} e^{-px-qy}f(x,y)\,dx\,dy,\quad R^{*}(p,q)>0$$

has been generalized by the author in the form [3]

$$(1.2)\qquad F(p,q)=pq\iint_0^{\infty} G^{m+1,0}_{m,m+1}\left(px\Big/{}^{(a_m+b_m)^{*}}_{(a_{m+1})}\right)G^{n+1,0}_{n,n+1}\left(qy\Big/{}^{(c_n+d_n)}_{(c_{n+1})}\right)$$
$$\times f(x,y)\,dx\,dy,\quad R(p,q)>0.$$

The Meijer-Laplace transform (1.2) will be denoted symbolically as $F(p,q)=G(f(x,y))$ whereas the Laplace transform (1.1) as $F(p,q)\doteqdot f(x,y)$. When $b_j=0$, $j=1,2,\cdots,m-1$; $d_i=0$, $i=1,2,\cdots,n-1$ and

(a) $b_m=a_{m+1}=d_n=c_{n+1}=0$, using $G^{1,0}_{0,1}(z/\bar{0})=e^{-z}$, (1.2) reduces to (1.1);

(b) $b_m=-m-k$, $a_m=m-k$, $a_{m+1}=-m-k$; $d_n=-m_1-k_1$, $c_n=m_1-k_1$, $c_{n+1}=-m_1-k_1$, (1.2) reduces to

$$(1.3)\qquad F(p,q)=pq\iint_0^{\infty} e^{-\frac{1}{2}px-\frac{1}{2}qy}(px)^{-k-\frac{1}{2}}(qy)^{-k_1-\frac{1}{2}}W_{k+\frac{1}{2},m}(px)$$
$$\times W_{k_1+\frac{1}{2},m_1}(qy)f(x,y)\,dx\,dy,\qquad R(p,q)>0$$

and is known as Meijer transform of two variables [4, p. 83];

(c) $a_m=2m$, $b_m=\frac{1}{2}-m-k$, $a_{m+1}=0$; $c_n=2m_1$, $d_n=\frac{1}{2}-m_1-k_1$, $c_{n+1}=0$, (1.2) reduces to

$$(1.4)\quad F(p,q)=pq\iint_0^{\infty} e^{-\frac{1}{2}px-\frac{1}{2}qy}(px)^{m-\frac{1}{2}}(qy)^{m_1-\frac{1}{2}}W_{k,m}(px)W_{k_1,m_1}(qy)$$

(*) For brevity, the symbol $\iint_0^{\infty}$ denotes $\int_0^{\infty}\int_0^{\infty}$, the symbol $R(p,q)>0$ denotes $R(p)>0$, $R(q)>0$ and (g_r) denotes the set of parameters $g_1, g_2, \cdots, g_r$.

$$\times f(x,y)dxdy, \quad R(p,q)>0,$$

which we shall call as Varma transform of two variables [5].

In this paper we have obtained a chain of Meijer-Laplace transform of two variables which yield interesting results in other transforms to which it reduces.

In what follows the symbol $\Delta(n,a)$ denotes the set of parameters $\frac{a}{n}$, $\frac{a+1}{n}$, ..., $\frac{a+n-1}{n}$, where n is a positive integer and the symbol $\Delta((n,a_r))$ denotes the set of parameters $\Delta(n,a_1)$, $\Delta(n,a_2)$, ..., $\Delta(n,a_r)$.

2. We shall require the following results which follow from the results given by Saxena [6, p. 401]:

(2.1)
$$\int_0^\infty t^{\sigma-1} G_{q,r}^{h,l}\left(pt \Big/ {(c_q) \atop (d_r)}\right) G_{\nu,\delta}^{\alpha,\beta}\left(zt^n \Big/ {(a_\nu) \atop (b_\delta)}\right) dt$$

$$= p^{-\sigma}(2\pi)^{(1-n)\left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)} n^{\Sigma_{i=1}^{r} d_i - \Sigma_{j=1}^{q} c_j + (\sigma-1/2)(r-q)}$$

$$\times G_{\nu+nr,\,\delta+nq}^{\alpha+nl,\,\beta+nh}\left(\frac{z}{p^n n^{n(q-r)}}\left| {(a_\beta),\,\Delta((n,-d_r-\sigma+1)),\,a_{\beta+1},\cdots,a_\nu \atop (b_\alpha),\,\Delta((n,-c_q-\sigma+1)),\,b_{\alpha+1},\cdots,b_\delta}\right.\right),$$

provided $R(p)>0, 0\leq nq\leq nr<nq+\delta-\nu; q+r<2h\leq 2r; 0\leq\beta\leq\nu, 1\leq\alpha\leq\delta; l=0,$
$R(\min d_i+n \min b_j)>R(-\sigma), i=1, 2, \cdots, h; j=1, 2, \cdots, \alpha;$
$|\arg p|<\left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi$ and $\arg(z)$ may have any value.

(2.2)
$$G_{0,2\alpha}^{2\alpha,0}\left(\left(\frac{ps}{2\alpha}\right)^{2\alpha} \Big/ {- \atop \Delta(2\alpha,1-2\alpha)}\right) = 2^{3\alpha-2}\pi^{\alpha-1/2}\alpha^{2\alpha-3/2}(ps)^{1-2\alpha}e^{-ps},$$

where α is a positive integer.

3. THEOREM. *If*

(3.1) $$F(p,q) = G(f_1(s,t)),$$

(3.2) $$(pq)^{\frac{1}{2}} f_1\left(\frac{1}{p}, \frac{1}{q}\right) = G(f_2(s,t)),$$

(3.3) $$\frac{\pi}{4}\left(\frac{1}{pq}\right)^{\frac{1}{2}} f_2\left(\frac{1}{4p^2}, \frac{1}{4q^2}\right) = G(f_3(s,t)),$$

(3.4) $$\frac{\pi}{4}\left(\frac{1}{pq}\right)^{\frac{1}{2}} f_3\left(\frac{1}{4p^2}, \frac{1}{4q^2}\right) = G(f_4(s,t)),$$

....

and

$$(3.5)\qquad \frac{\pi}{4}\left(\frac{1}{pq}\right)^{\frac{1}{2}} f_{r-1}\left(\frac{1}{4p^2}, \frac{1}{4q^2}\right) = G(f_r(s,t)),$$

then

$$(3.6)\qquad F(p,q)=2^{3r+2\alpha-4r\alpha-4}\pi^{2-2\alpha}\prod_{2}^{r}(\alpha^{a_{m+1}+c_{n+1}-\Sigma_{i=1}^{m}b_i-\Sigma_{j=1}^{n}d_j})$$

$$\times pq\int_0^\infty\int G^{2\alpha(m+1),0}_{2\alpha m,2\alpha(m+1)}\left(\frac{ps^{2\alpha}}{(2\alpha)^{2\alpha}}\Big/\begin{matrix}\Delta\left(\left(\alpha,a_m+b_m-\frac{2\alpha-1}{2}\right)\right),\cdots,\Delta\left(\left(1,a_m+b_m-\frac{1}{2}\right)\right),(a_m+b_m)\\ (a_{m+1}),\ \Delta\left(\left(1,a_{m+1}-\frac{1}{2}\right)\right),\cdots,\Delta\left(\left(\alpha,a_{m+1}-\frac{2\alpha-1}{2}\right)\right)\end{matrix}\right)$$

$$\times G^{2\alpha(n+1),0}_{2\alpha n,2\alpha(n+1)}\left(\frac{qt^{2\alpha}}{(2\alpha)^{2\alpha}}\Big/\begin{matrix}\Delta\left(\left(\alpha,c_n+d_n-\frac{2\alpha-1}{2}\right)\right),\cdots,\Delta\left(\left(1,c_n+d_n-\frac{1}{2}\right)\right),(c_n+d_n)\\ (c_{n+1}),\ \Delta\left(\left(1,c_{n+1}-\frac{1}{2}\right)\right),\cdots,\Delta\left(\left(\alpha,c_{n+1}-\frac{2\alpha-1}{2}\right)\right)\end{matrix}\right)$$

$$\times s^{2\alpha}t^{2\alpha}f_r\left(\frac{s^2}{4}, \frac{t^2}{4}\right)dsdt,$$

provided $R(p,q)>0$, the Meijer-Laplace transform of $|f_k(s,t)|$ for $k=1,2,\cdots,r$ all exist, and the integrals involved are absolutely convergent. Here $\alpha=2^{r-2}$ and $\prod_{2}^{r}$ indicates the product of the factors within bracket for $r=2$ to any integral value of r.

PROOF. Substituting the value of $f_1(s,t)$ from (3.2) in (3.1), interchanging the order of double integration which is permissible due to absolute convergence of the integrals, using [1, p. 209, (9)] and evaluating the later double integral with the help of (2.1), replacing s by $\frac{s^2}{4}$ and t by $\frac{t^2}{4}$, we get

$$(3.7)\qquad F(p,q)=2^{-4}pq\int_0^\infty\int G^{2(m+1),0}_{2m,\,2(m+1)}\left(\frac{ps^2}{4}\Big/\begin{matrix}\Delta\left(\left(1,a_m+b_m-\frac{1}{2}\right)\right),(a_m+b_m)\\ (a_{m+1}),\ \Delta\left(\left(1,a_{m+1}-\frac{1}{2}\right)\right)\end{matrix}\right)$$

$$\times G^{2(n+1),0}_{2n,2(n+1)}\left(\frac{qt^2}{4}\Big/\begin{matrix}\Delta\left(\left(1,c_n+d_n-\frac{1}{2}\right)\right),\ (c_n+d_n)\\ (c_{n+1}),\ \Delta\left(\left(1,c_{n+1}-\frac{1}{2}\right)\right)\end{matrix}\right)(st)^2 f_2\left(\frac{s^2}{4}, \frac{t^2}{4}\right)dsdt.$$

From (3.3) putting the value of $f_2\left(\frac{s^2}{4}, \frac{t^2}{4}\right)$ in (3.7) and proceeding as above, we get

$$(3.8)\qquad F(p,q)=2^{a_{m+1}+c_{n+1}-\Sigma_{i=1}^{m}b_i-\Sigma_{j=1}^{n}d_j-15}\pi^{-2}\cdot pq$$

$$\times\int_0^\infty\int G^{4(m+1),0}_{4m,\,4(m+1)}\left(\frac{ps^4}{4^4}\Big/\begin{matrix}\Delta\left(\left(2,a_m+b_m-\frac{3}{2}\right)\right),\ \Delta\left(\left(1,a_m+b_m-\frac{1}{2}\right)\right),(a_m+b_m)\\ (a_{m+1}),\ \Delta\left(\left(1,a_{m+1}-\frac{1}{2}\right)\right),\ \Delta\left(\left(2,a_{m+1}-\frac{3}{2}\right)\right)\end{matrix}\right)$$

$$\times G^{4(n+1),0}_{4n,4(n+1)}\left(\frac{qt^4}{4^4}\middle/ \begin{matrix} \Delta\left(\left(2, c_n+d_n-\frac{3}{2}\right)\right), \Delta\left(\left(1, c_n+d_n-\frac{1}{2}\right)\right), (c_n+d_n) \\ (c_{n+1}), \Delta\left(\left(1, c_{n+1}-\frac{1}{2}\right)\right), \Delta\left(\left(2, c_{n+1}-\frac{3}{2}\right)\right) \end{matrix}\right)$$

$$\times (st)^4 f_3\left(\frac{s^2}{4}, \frac{t^2}{4}\right) ds dt$$

Repeating this process successively with correspondences (3.4), ……, we arrive at the result.

COROLLARY. Taking $b_j=0$, $j=1,2,\cdots,m$, $a_{m+1}=0$; $d_i=0$, $i=1,2,\cdots,n$, $c_{n+1}=0$, simplifying and then using (2.2), we get the following chain in Laplace transform of two variables.

If

$$F(p,q) \doteqdot f_1(s,t),$$

$$(pq)^{-\frac{1}{2}} f_1\left(\frac{1}{p}, \frac{1}{q}\right) \doteqdot f_2(s,t),$$

$$\frac{\pi}{4}\left(\frac{1}{pq}\right)^{\frac{1}{2}} f_2\left(\frac{1}{4p^2}, \frac{1}{4q^2}\right) \doteqdot f_3(s,t),$$

$$\frac{\pi}{4}\left(\frac{1}{pq}\right)^{\frac{1}{2}} f_3\left(\frac{1}{4p^2}, \frac{1}{4q^2}\right) \doteqdot f_4(s,t),$$

……………………………………………

and

$$\frac{\pi}{4}\left(\frac{1}{pq}\right)^{\frac{1}{2}} f_{r-1}\left(\frac{1}{4p^2}, \frac{1}{4q^2}\right) \doteqdot f_r(s,t),$$

then

$$F(p^{2^{r-1}}, q^{2^{r-1}}) \doteqdot \frac{\pi st}{4} f_r\left(\frac{s^2}{4}, \frac{t^2}{4}\right),$$

provided $R(p,q)>0$, and the Laplace transform of $|f_i(s,t)|$ and $\left|st\, f_i\left(\frac{s^2}{4}, \frac{t^2}{4}\right)\right|$ for $i=2,3,\cdots,r$ all exist.

Putting $b_j=0$, $j=1,\cdots, m-1$, $b_m=-m-k$, $a_m=m-k$, $a_{m+1}=-m-k$, $d_i=0$, $i=1,\cdots, n-1$, $d_n=-m_1-k_1$, $c_n=m_1-k_1$, $c_{n+1}=-m_1-k_1$, in (3.1), (3.2), $\cdots$, we get a corresponding chain of Meijer transform of two variables.

Setting $b_j=0$, $j=1,2,\cdots, m-1$, $b_m=\frac{1}{2}-m-k$, $a_m=2m$, $a_{m+1}=0$; $d_i=0$, $i=1$, $\cdots, n-1$, $d_n=\frac{1}{2}-m_1-k_1$, $c_n=2m_1$, $C_{n+1}=0$ in (3.1), (3.2), ……, we get a chain of Varma transform of two variables.

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