A GENERALIZATION OF A THEOREM OF ALFSEN AND FENSTAD

By C. J. Mozzochi

In this paper the theorem of Alfsen and Fenstad, namely that every proximity class of uniform spaces contains one and only one totally bounded uniform space, is generalized to symmetric generalized uniform spaces (introduced by the author in [2]). Also, a new characterization of totally bounded uniform spaces is obtained.

This paper is based on part V of the author's thesis, *Symmetric generalized uniform and proximity spaces*, submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Graduate School of Arts and Sciences of the University of Connecticut. The author wishes to acknowledge his indebtedness to Professor E. S. Wolk, under whose direction the thesis was written.

Let X be a non-void set. For every A, B in $P(X)$ let $U_{A,B}=(X\times X)-((A\times B)\cup(B\times A))$.

DEFINITION. Let $(X,\mathscr{U})$ be a symmetric generalized uniform space. $(X,\mathscr{U})$ is *p-correct* iff there exists a symmetric generalized proximity δ on X such that the family $\mathscr{S}=\{U_{A,B}|A\ \bar{\delta}\ B\}$ is a subbase for $\mathscr{U}$. δ is called the *generator proximity* for $\mathscr{U}$.

LEMMA 1. *Let* $(A_1, \cdots, A_n)$ *and* $(B_1, \cdots, B_n)$ *be n-tuples of non-void subsets of a set X. Let* $U=U_{A_1,B_1}\cap\cdots\cap U_{A_n,B_n}$. *Let* $I_1=\{k_1, \cdots, k_p\}$ *and* $I_2=\{j_1, \cdots, j_q\}$ *be subsets of* $\{1, \cdots, n\}$. *Suppose* $x_0 \in (A_{k_1}\cap\cdots\cap A_{k_p}\cap B_{j_1}\cap\cdots\cap B_{j_q})$ *and* $x_0\notin A_i$ *if* $i\notin I_1$ *and* $x_0\notin B_i$ *if* $i\notin I_2$. *Then* $U[x_0]=E$, *where E is equal to*

$$(X-B_{k_1})\cap\cdots\cap(X-B_{k_p})\cap(X-A_{j_1})\cap\cdots\cap(X-A_{j_q}).$$

REMARK. In the sequel to simplify the language we will abbreviate the hypothesis of Lemma 1 as follows: "Suppose $x_0\in(A_{k_1}\cap\cdots\cap A_{k_p}\cap B_{j_1}\cap\cdots\cap B_{j_q})$ and x_0 is in no other A_i or B_i."

PROOF of LEMMA 1. By De Morgan's law

$$U=(X\times X)-(\bigcup_{i=1}^{n}[(A_i\times B_i)\cup(B_i\times A_i)]).$$

Suppose $t\in U[x_0]$. Then $(x_0, t)\in U$; so that since $x_0 \in (A_{k_1}\cap\cdots\cap A_{k_p}\cap B_{j_1}\cap\cdots\cap B_{j_q})$

we have that $t \notin B_{k_i}$ $i=1, \cdots, p$ and $t \notin A_{j_i}$ $i=1, \cdots, q$. Consequently, $t \in E$ and $E \supset U[x_0]$. To show the reverse inclusion, suppose there exists $t_1 \in (E-U[x_0])$. Then $(x_0, t_1) \notin U$; so that (x_0, t_1) is an element of $\bigcup_{i=1}^{n} [(A_i \times B_i) \cup (B_i \times A_i)]$. Suppose $(x_0, t_1) \in (A_m \times B_m)$ where $1 \leq m \leq n$. Then since $t_1 \in E$, we have that $m \neq k_i$ for $i=1, \cdots, p$; so that $x_0 \in A_m$ and $m \notin I_1$ which is a contradiction. Suppose $(x_0, t_1) \in (B_m \times A_m)$ where $1 \leq m \leq n$. Then since $t_1 \in E$, we have that $m \neq j_i$ for $i=1, \cdots, q$; so that $x_0 \in B_m$ and $m \notin I_2$ which is a contradiction. Hence $E=U[x_0]$.

REMARK. Let $(A_1, \cdots, A_n)$ and $(B_1, \cdots, B_n)$ be n-tuples of non-void subsets of a set X. $I_1=\{k_1, \cdots, k_p\}$ and $I_2=\{j_1, \cdots, j_q\}$ be any two subsets of $\{1, \cdots, n\}$ and let
$E=\{x \mid x \in A_i$ *iff* $i \in I_1$ *and* $x \in B_i$ *iff* $i \in I_2\}$. If $E \neq \phi$, we call E a *residual intersection of the* A_i *and* B_i.

It is clear that residual intersections are mutually disjoint; so that $\mathscr{R}$, the family of all residual intersections of the A_i and B_i, provides a decomposition of $\bigcup\{(A_i \cup B_i) \mid i=1, \cdots, n\}$ into mutually disjoint sets.

THEOREM 2. *Let* $(X, \mathscr{U})$ *be a* p*-correct symmetric generalized uniform space. Then* $(X, \mathscr{U})$ *is totally bounded.*

PROOF. Let $U \in \mathscr{U}$, and let δ be a generator proximity for $\mathscr{U}$. Then there exists a finite family of sets $A_1, \cdots, A_n; B_1, \cdots, B_n$ such that $A_i \bar{\delta} B_i$ for $i=1, \cdots, n$ and $U_{A_1, B_1} \cap \cdots \cap U_{A_n, B_n} = V \subset U$. Now if $\bigcup\{(A_i \cup B_i) \mid i=1, \cdots, n\} \neq X$, then for any $x_0 \in X - \bigcup\{(A_i \cup B_i) \mid i=1, \cdots, n\}$ we have that $V[x_0]=X$, and the theorem follows; so we assume that $\bigcup\{(A_i \cup B_i) \mid i=1, \cdots, n\}=X$. Let $\mathscr{R}$ be the family of all residual intersections of the A_i and B_i. From each $R \in \mathscr{R}$ choose one and only one point and denote that point x_R. Let $S=\{x_R \mid R \in \mathscr{R}\}$. Clearly, since $\mathscr{R}$ is finite, S is also finite. We now show that $V[S]=X$. Let $z \in X$. Since we assume that $\bigcup\{(A_i \cup B_i) \mid i=1, \cdots, n\}=X$, we have that $z \in R$ for some $R \in \mathscr{R}$. Consequently, for some $k_1, \cdots, k_p; j_1, \cdots, j_q$, $z \in (A_{k_1} \cap \cdots \cap A_{k_p} \cap B_{j_1} \cap \cdots \cap B_{j_q})$ and z is in no other A_i or B_i. But by the definition of S there exists x_R in S such that $x_R \in (A_{k_1} \cap \cdots \cap A_{k_p} \cap B_{j_1} \cap \cdots \cap B_{j_q})$ and x_R is in no other A_i or B_i. By Lemma 1 we have that $V[x_R]$ is equal to $(X-B_{k_1}) \cap \cdots \cap (X-B_{k_p}) \cap (X-A_{j_1}) \cap \cdots \cap (X-A_{j_q})$. But since $A_i \bar{\delta} B_i$ for all i we have that $z \notin B_{k_i}$ for $i=1, \cdots, p$ and $z \notin A_{j_i}$ for $i=1, \cdots, q$. Consequently, $z \in V[x_R]$.

But z is an arbitrary point in X. Hence $V[S]=X$; so that $U[S]=X$.

THEOREM 3. *A symmetric uniform space $(X, \mathscr{U})$ is totally bounded iff for some proximity δ on X the family $\mathscr{S}=\{U_{A,B} | A \bar{\delta} B\}$ is a subbase for $(X, \mathscr{U})$.*

LEMMA 4. *Suppose $\{A_i\}$ and $\{B_i\}$, $i=1, \cdots, n$ are finite sequences of non-void subsets of a set X such that for all i $A_i \supset B_i$ and $\cup\{B_i | i=1, \cdots, n\}=X$. Then we have that*

$$F=(X\times X)-\bigcup_{i=1}^{n}\Big[\,[(X-A_i)\times B_i] \cup [B_i\times(X-A_i)]\,\Big]\subset \bigcup_{i=1}^{n} [A_i\times A_i].$$

PROOF of LEMMA 4. Let $(x,y)\in F$. Then since $\cup\{B_i | i=1, \cdots, n\}=X$ we have that $(x, y)\in (B_{k_1}\times B_{k_2})$ where $1\leq k_1\leq n$ and $1\leq k_2\leq n$. But it is clear that $(x, y)\notin [(X-A_{k_2})\times B_{k_2}]$; so that since $y\in B_{k_2}$, $x\in A_{k_2}$. But $A_{k_2}\supset B_{k_2}$. Hence $(x, y)\in (A_{k_2}\times A_{k_2})$.

LEMMA 5. *Let (X, δ) be a proximity space. Let $\mathscr{U}$ be a totally bounded symmetric uniformity on X that is in $\pi^*(\delta)$, a proximity class of symmetric uniformities on X. Then for every $U\in\mathscr{U}$ there exist sets $A_1, \cdots, A_n; B_1, \cdots, B_n$ uch that $U\supset U_{A_1,B_1}\cap\cdots\cap U_{A_n,B_n}$* and $A_i \bar{\delta} B_i$ for $i=1, \cdots, n$.

PROOF of LEMMA 5. Let $U\in\mathscr{U}$. We know there exists $V\in\mathscr{U}$ such that $V=V^{-1}$ and $(V\circ V\circ V)\subset U$. Then since $(X, \mathscr{U})$ is totally bounded, there exist sets $B_1, \cdots, B_n$ such that $\bigcup_{i=1}^{n}[B_i]=X$ and $\bigcup_{i=1}^{n}[B_i\times B_i]\subset V$. Let $A_i=V[B_i]$. Since $V[B_i]\cap (X-V[B_i])=\phi$, $A_i \gg B_i$, $i=1, \cdots, n$. Also, by a straightforward calculation, we can show for $i=1, \cdots, n$ that $(A_i\times A_i)\subset V\circ V\circ V$. Hence we have that $\bigcup_{i=1}^{n}[A_i\times A_i]\subset U$. But by Lemma 4

$$(X\times X)-\bigcup_{i=1}^{n}\Big[\,[(X-A_i)\times B_i]\cup[B_i\times(X-A_i)]\,\Big]\subset \bigcup_{i=1}^{n}[A_i\times A_i];$$

so that
$$U_{B_1,X-A_1}\cap\cdots\cap U_{B_n,X-A_n}\subset U,$$
and
$$B_i \bar{\delta} (X-A_i) \text{ for } i=1, \cdots, n.$$

PROOF of THEOREM 3. Suppose for some proximity δ on X $\mathscr{S}=\{U_{A,B} | A \bar{\delta} B\}$ is a subbase for $\mathscr{U}$. Then $\mathscr{U}$ is a p-correct symmetric generalized uniformity on X, and hence by Theorem 2 $\mathscr{U}$ is totally bounded.

Conversely, suppose $\mathscr{U}$ is totally bounded. It is known (cf. [3] Theorem (21.14) and Theorem (21.15)) that for some proximity δ on X $\mathscr{U}\in\pi^*(\delta)$, a proximity class of symmetric uniformities on X. Suppose $A_i \bar{\delta} B_i$ for $i=1, \cdots, n$. For each i, $i=1, \cdots, n$ there exists a symmetric $V_i\in\mathscr{U}$ such that

$(A_i \times B_i) \cap V_i = \phi$, and hence such that $U_{A_i, B_i} \supset V_i$. Consequently, we have that $U = (U_{A_1, B_1} \cap \cdots \cap U_{A_n, B_n}) \supset (V_1 \cap \cdots \cap V_n)$; so that $U \in \mathscr{U}$. By this fact and Lemma 5 we have that the family $\mathscr{S} = \{U_{A,B} | A \bar{\delta} B\}$ is a subbase for $\mathscr{U}$.

THEOREM 6. *Let (X, δ) be a symmetric generalized proximity space. There exists in $\pi(\delta)$ one and only one p-correct symmetric generalized uniformity, $\mathscr{U}_2(\delta)$, on X.*

LEMMA 7. *Let (X, δ) be a symmetric generalized proximity space. Let $(C_1, \cdots, C_n)$ and $(D_1, \cdots, D_n)$ be n-tuples of non-void subsets of X such that $C_i \bar{\delta} D_i$ for $i=1, \cdots, n$. Then $(C_1 \cap \cdots \cap C_n) \bar{\delta} (D_1 \cup \cdots \cup D_n)$.*

PROOF of LEMMA 7. Suppose that $(C_1 \cap \cdots \cap C_n) \delta (D_1 \cap \cdots \cap D_n)$. Then $(C_1 \cap \cdots \cap C_n) \delta D_k$ where $1 \leq k \leq n$. But $C_k \supset (C_1 \cap \cdots \cap C_n)$; so that $C_k \delta D_k$ which is a contradiction.

LEMMA 8. *Let (X, δ) be a symmetric generalized proximity space. Then $P \bar{\delta} Q$ iff there exist n-tuples $(A_1, \cdots, A_n)$ and $(B_1, \cdots, B_n)$ of subsets of X such that $(U_{A_1, B_1} \cap \cdots \cap U_{A_n, B_n})[P] \cap Q = \phi$, and $A_i \bar{\delta} B_i$ for $i=1, \cdots, n$.*

PROOF of LEMMA 8. If $P \bar{\delta} Q$, then it is clear that $U_{P,Q}[P] \cap Q = \phi$.

Conversely, let $V = U_{A_1, B_1} \cap \cdots \cap U_{A_n, B_n}$. Since $V[P] \cap Q = \phi$, we have $P \subset \cup \{(A_i \cup B_i) | i=1, \cdots, n\}$. Let $\mathfrak{A} = \{E_1, \cdots, E_m\}$ be the pairwise disjoint family of all residual intersections of the A_i and B_i that have a non-void intersection with P. Clearly, $P \subset M = \cup \{E_c | c=1, \cdots, m\}$. By Lemma 1 since $\mathfrak{A}$ is a pairwise disjoint family, if $t_1 \in (P \cap E_c)$ and $t_2 \in (P \cap E_c)$ where $1 \leq c \leq m$, then $V[t_1] = V[t_2]$. Let $F_c = V[t_c]$ for $c=1, \cdots, m$ where t_c is a fixed point in E_c. Then we have that $V[P] = \cup \{F_c | c=1, \cdots, m\}$. But since $V[P] \cap Q = \phi$ we have that $Q \subset (X - V[P])$; so that by De Morgan's law $Q \subset N$ where $N = \cap \{(X - F_c) | c=1, \cdots, m\}$. Let $E_c \in \mathfrak{A}$ where $1 \leq c \leq m$. We may assume that $E_c \subset E_c^* = A_{k_1} \cap \cdots \cap A_{k_p} \cap B_{j_1} \cap \cdots \cap B_{j_q}$ for some $k_1, \cdots, k_p$; $j_1, \cdots, j_q$ and E_c intersects no other A_i or B_i. Consequently, by Lemma 1 and De Morgan's law $(X - F_c) = (B_{k_1} \cup \cdots \cup B_{k_p} \cup A_{j_1} \cup \cdots \cup A_{j_q})$. Hence by Lemma 7 $E_c^* \bar{\delta} (X - F_c)$ where $1 \leq c \leq m$; so that $E_c \bar{\delta} (X - F_c)$ where $1 \leq c \leq m$. Hence again by Lemma 7 $M \bar{\delta} N$; so that $P \bar{\delta} Q$.

LEMMA 9. *Let $(X, \mathscr{U})$ be a p-correct symmetric generalized uniform space with generator proximity δ. Then $\delta(\mathscr{U}) = \delta$.*

PROOF of LEMMA 9. Suppose $P \bar{\delta} Q$. Then by Lemma 8 there exists

$U \in \mathscr{U}$ such that $U[P] \cap Q = \phi$; so that $P \overline{\delta(\mathscr{U})} Q$.

Conversely, suppose $P \overline{\delta(\mathscr{U})} Q$. Then there exists $V \in \mathscr{U}$ such that $V[P] \cap Q = \phi$; so that by Lemma 8 $P \bar{\delta} Q$.

PROOF of THEOREM 6. For the notation used in this proof see [2]. Let $\mathscr{S} = \{U_{A,B} | A \bar{\delta} B\}$. Let $\mathscr{B} = \{$*all finite intersections of members of* $\mathscr{S}\}$. It is clear that $\mathscr{B}$ satisfies (M.1) and (M.2). By Lemma 8 and Theorem 1 in [2] we have that $\mathscr{B}$ also satisfies (M.3) *and* (M.4). Consequently, by Theorem (5) in [2] we have that $\mathscr{U}_2(\delta) = \{U | U = U^{-1}$ *and* $V \supset U$ *for some* $V \in \mathscr{B}\}$ is a symmetric generalized uniformity on X. It is clear that $\mathscr{U}_2(\delta)$ is p-correct, and by Lemma 9 that $\mathscr{U}_2(\delta) \in \pi(\delta)$. We now show that $\mathscr{U}_2(\delta)$ is the only p-correct symmetric generalized uniformity on X that is in $\pi(\delta)$. For suppose $\mathscr{V} \in \pi(\delta)$ and $(X, \mathscr{V})$ is p-correct with generator proximity δ_1. Clearly, $\delta_1 \neq \delta$ if $\mathscr{U}_2(\delta) \neq \mathscr{V}$. But by Lemma 9 we have that $\delta(\mathscr{V}) = \delta_1$ which is a contradiction, since we assume $\mathscr{V} \in \pi(\delta)$. Hence $\mathscr{V} = \mathscr{U}_2(\delta)$.

COROLLARY 10. (Alfsen-Fenstad). *Let* (X, δ) *be a proximity space. There exists in* $\pi(\delta)$ *one and only one totally bounded symmetric uniformity on* X.

PROOF. By Theorem 3 and Theorem 6, it is sufficient to show that $\mathscr{U}_2(\delta)$ satisfies the triangle axiom. We note that if $V_i \circ V_i \subset U_i$ for $i=1, \cdots, n$, then $(V_1 \cap \cdots \cap V_n) \circ (V_1 \cap \cdots \cap V_n) \subset U_1 \cap \cdots \cap U_n$ where V_i and U_i for $i=1, \cdots, n$ are subsets of $(X \times X)$. Consequently, it is sufficient to show that for each $U_{A,B} \in \mathscr{U}_2(\delta)$ there exist a $V \in \mathscr{U}_2(\delta)$ such that $V \circ V \subset U_{A,B}$. We now show the existence of such a V. Since $A \bar{\delta} B$ there exist sets C and D such that $C \cap D = \phi$ and $C \gg A$ and $D \gg B$. Let $V = (U_{A, X-C}) \cap (U_{B, X-D})$. We show $V \circ V \subset U_{A,B}$. Suppose $(x, y) \in V$ and $(y, z) \in V$. We must show that $(x, z) \in U_{A,B}$ or equivalently that $(x, z) \notin (A \times B) \cup (B \times A)$. Clearly, if $x \notin (A \cup B)$, then for every $t \in X$ we have that $(x, t) \in U_{A,B}$. Hence we may assume that $x \in (A \cup B)$. Two cases now occur. Case 1, $x \in A$, and Case 2, $x \in B$. These are the only possibilities for x since $A \cap B = \phi$.

CLAIM 1. If $x \in A$, then $z \notin B$. For suppose $z \in B$. Then $(y, z) \in (C \times B)$. But since $C \cap D = \phi$, $(X-D) \supset C$; so that $((X-D) \times B) \supset (C \times B)$. Hence $(y, z) \notin V$ which is a contradiction. By a similar argument we get

CLAIM 2. If $x \in B$, then $z \notin A$.

By claim 1 if $x \in A$, then $(x, z) \notin (A \times B)$; so that $(x, z) \in U_{A,B}$. By Claim 2 if $x \in B$, then $(x, z) \notin (B \times A)$; so that $(x, z) \in U_{A,B}$.

Trinity College
Hartford, Connecticut
U.S.A.

REFERENCES

[1] E.M. Alfsen and J.E. Fenstad; *On the equivalence between proximity structures and totally bounded uniform structures,* Math. Scand. 7 (1959), 353-360.

[2] C.J. Mozzochi; *A correct system of axioms for a symmetric generalized uniform space* (to appear) Math. Scand.

[3] W.J. Tihron; *Topological structures,* Holt, Rinehart, and Winston, New York. 1966.