J. Korean Math. Soc. **6** (1969), 81—83

On the irreducible closed sets of special structure spaces of rings

SEUNG AHN PARK

1. Introduction. Let k be a field and $\bar{k}$ an algebraic closure of k. By an *algebraic set in* $\bar{k}^n$ we mean the zeros of an ideal of $k[X_1, X_2, \cdots\cdots, X_n]$. An algebraic set is called *irreducible* if it cannot be expressed as a proper union of two algebraic sets. Then we have a decomposition of an algebraic set into irreducible algebraic sets, and this decomposition is unique if there is no inclusion relation among the irreducible algebraic sets. Also, an algebraic set is an irreducible algebraic set if and only if its associated ideal in $k[X]$ is a prime ideal, and by the Hilbert Nullstellensatz, there is a 1-1 correspondence between prime ideals I of $k[X]$ and irreducible algebraic sets C, given by C= zeros of I and I= ideal vanishing on C.

The notion of algebraic set can be generalized to arbitrary commutative ring with an identity. Thus, for a commutative ring K with an identity let $S_\Sigma(K)$ be the set of proper prime ideals of K. Introducing Zariski topology on $S_\Sigma(K)$, we can generalize the notion of algebraic set to the notion of closed set of $S_\Sigma(K)$. [3, pp. 259-260]

Our main goal in this paper is to show that the notion of algebraic set can be generalized to the notion of closed set of Σ-special structure spaces of an arbitrary ring.

2. Main theorems. In this section, K denotes a non-commutative ring.

A class of rings Σ is said to be *special* if:

1) every ring in Σ is a prime ring,
2) any non-zero ideal of a ring of Σ belongs to Σ,
3) if K is a prime ring and $A \varepsilon \Sigma$ is one of its nonzero ideals, then $K \varepsilon \Sigma$.

We will denote by R_Σ the special radical determined by the special class Σ, and by $S_\Sigma(K)$ the set of Σ-special ideals of the ring K, *i.e.*, those ideals I such that $K/I \varepsilon \Sigma$. The set $S_\Sigma(K)$ with the topology in which the closure of a subset $M \subseteq S_\Sigma(K)$ is defined as

$$\overline{M}=\{I \mid I \varepsilon S_\Sigma(K),\ I \supseteq D_M\} \quad \text{where } D_M=\bigcap_{B \varepsilon M} B$$

is called *the Σ-special structure space of the ring* K. If U is a subset of the ring K, we define $Q_\Sigma(U)$ by

$$Q_\Sigma(U)=\{I;\ I \varepsilon S_\Sigma(K),\ I \supseteq U\}. \quad [1]$$

Presented to the Society on October 24, 1969; received by the editors on October 31, 1969.

PROPOSITION. *If K is a prime ring and there exist nonzero ideal A which is a simple ring with an identity (resp. a primitive ring), then K is a simple ring with an identity (resp. a primitive ring).*

Proof: By the assumption, $\{0\}$ is a modular maximal ideal (resp. a primitive ideal) of the ring A. Hence $(0:A)$ is a modular maximal ideal (resp. a primitive ideal) of K. [2, p. 206.] Since $A(0:A)=\{0\}$ and K is a prime ring, $(0:A)=\{0\}$, and the proof is completed.

By this result, the class Σ of simple rings with identities (resp. primitive rings, prime rings) is special, and therefore the set $S_\Sigma(K)$ of modular maximal ideals (resp. primitive ideals, proper prime ideals) of the ring K with the topology defined as above is Σ-special structure space of K.

DEFINITION. A closed set C in $S_\Sigma(K)$ is called ***irreducible*** if it can not be expressed as a proper union of two closed sets, *i.e.*, if $C \neq C_1 \cup C_2$, with C_1 and C_2 distinct from C.

THEOREM 1. *Any clobed set C in $S_\Sigma(K)$ is irreducible if and only if its associated ideal D_C is a prime ideal of K.*

Proof: Let C be irreducible, and suppose that U_1 and U_2 are ideals of K such that $U_1U_2 \subseteq D_C$. Since any Σ-special ideal is a prime ideal of K,

$$C \subseteq Q_\Sigma(U_1U_2) = Q_\Sigma(U_1) \cup Q_\Sigma(U_2).$$

Hence

$$C = (C \cap Q_\Sigma(U_1)) \cup (C \cap Q_\Sigma(U_2)).$$

And, each term of the right-hand side of this equality is closed sets and so we have $C \cap Q_\Sigma(U_1) = C$ or $C \cap Q_\Sigma(U_2) = C$. Therefore $C \subseteq Q_\Sigma(U_1)$ or $C \subseteq Q_\Sigma(U_2)$ which implies $D_C \supseteq U_1$ or $D_C \supseteq U_2$. Hence D_C is a prime ideal of K.

Conversely, suppose $C = A \cup B$ where A and B are closed sets. Then $D_C = D_{A \cup B} = D_A \cap D_B \supseteq D_A D_B$ and so $D_C \supseteq D_A$ or $D_C \supseteq D_B$. Consequently $C \subseteq A$ or $C \subseteq B$, which implies C is irreducible.

Even if C is an irreducible closed set in the Σ-special structure space of modular maximal ideals (resp. of primitive ideals), the associated ideal D_C is not a modular maximal ideal (resp. a primitive ideal), in general. For example, let J be the ring of integers. Then, $S_\Sigma(J) = \{(p) \mid p \text{ is a prime number}\}$ is an irreducible closed set. Thus, if $S_\Sigma(J) = A \cup B$, then one of A and B, say A, is infinite set and so $D_A = \{0\}$; hence $A = S_\Sigma(J)$. Therefore, $S_\Sigma(J)$ is irreducible. However, $D_S = \{0\}$ is neither modular maximal nor primitive.

COROLLARY 1. *If the special radical R_Σ is a prime ideal, then the Σ-special structure space is connected.*

COROLLARY 2. *Let $S_\Sigma(K)$ be the set of proper prime ideals of K, then there is* *a* 1-1 *correspondence between prime ideals I in K and irreducible closed set C in $S_x(K)$, given by $C=Q_x(I)$ and $I=D_C$.*

If C and E are closed sets, then it is clear that $C \subseteq E$ if and only if $D_C \supseteq D_E$. Hence $C=E$ if and only if $D_C=D_E$. Therefore, if a ring K satisfies the maximum condition for (two-sided) ideals, then closed sets in $S_x(K)$ satisfy the descending chain condition. Using this and repeating the corresponding argument for algebraic sets [**3**, p. 258] we can easily verify the following

THEOREM 2. *Let K be a ring satisfying the maximum condition for ideals. Then every closed set C can be expressed as a finite union of irreducible closed sets; $C=C_1 \cup \cdots\cdots \cup C_r$. If there is no inclusion relation among the C_i, i.e., if*

$C_i \nsubseteq C_j$ $(i \neq j)$,

then this representation is unique.

REMARK. The author owes to Professor Wuhan Lee for his suggestions in the preparation of this manuscript.

References

1. Mihalev, A. V., *Special structure spaces of rings*, Soviet Math., Doklady, Amer. Math. Soc., **4** (1963), 651—653.
2. Nathan Jacobson, *Structure of Rings*, Amer. Math. Soc., Providence, 1964.
3. Serge Lang, *Algebra*, Addison-Wesley, New York, 1965.

Sogang Jesuit College