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Integral estimate for an elliptic system of partial differential operators with mixed boundary operators

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1. Introduction Let G be a bounded domain of class C^∞ in E_n. We denote its closure by $\bar{G}$ and its boundary by ∂G. Further, let Π be an n-2 dimensional manifold of class C^∞ on ∂G which divides ∂G into two components $\partial^{(1)}G$ and $\partial^{(2)}G$. A point of $\bar{G}$ is denoted by $x=(x_1, \cdots\cdots, x_n)$. We set

$$D=\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots\cdots\cdots\cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\right),$$

$$D^\mu=\left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\mu_1}\cdots\cdots\cdots\cdots\left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\mu_n},$$

where $\mu=(\mu_1, \cdots, \mu_n)$ is a multi-index and $|\mu|=\mu_1+\cdots+\mu_n$ is the order of the differential operator D^μ. Consider in $\bar{G}$, a system of differential operators

$$(1.1)\qquad \mathfrak{A}(x, D)=\begin{pmatrix} a_1(x, D) & & 0 \\ & \ddots & \\ 0 & & a_N(x, D)\end{pmatrix}$$

$$a_k(x, D)=\sum_{|\mu|\leqq 2m}\alpha_\mu^{(k)}(x)D^\mu, \quad k=1, \cdots, N,$$

$\alpha_\mu^{(k)}(x)$ being of class $C^\infty(\bar{G})$ together with systems of boundary operators

$$(1.2)\qquad \mathscr{B}^{(i)}(x, D)=(b_{j,k}^{(i)}(x, D)),$$

$$b_{j,k}^{(i)}(x, D)=\sum_{|\mu|\leqq 2m-\nu_j^{(i)}}\beta_\mu^{(i,j,k)}(x)D^\mu, \quad \nu_j^{(i)}\geqq 1,$$

$$i=1,2;\ j=1, \cdots, Nm;\ k=1, \cdots, N.$$

We assume that all the $\beta_\mu^{(i,j,k)}(x)$ are of class $C^\infty(\bar{G})$. And let

$$(1.3)\qquad \mathfrak{A}'(x, D)=\begin{pmatrix} a_1'(x, D) & & \\ & \ddots & \\ & & a'_N(x, D)\end{pmatrix}$$

$$(1.4)\qquad \mathscr{B}^{(i)\prime}(x, D)=(b_{j,k}^{(i)\prime}(x, D)),$$

$i=1, 2;\ j=1, \cdots, Nm;\ k=1, \cdots, N$, denote the characteristic parts of the operators (1.1) and (1.2) respectively. The characteristic matrices of (1.1), (1.2) are denoted by $\mathfrak{A}'(x, \xi)$, $\mathscr{B}^{(1)\prime}(x, \xi)$ and $\mathscr{B}^{(2)\prime}(x, \xi)$.

Now, our aim is to give an integral estimate for the system (1.1) and (1.2)

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under the following assumptions I—IV (c.f. Theorem 3.2). Applications of this estimate to the existence theorems for the corresponding differential equations will be discussed in the future.

(I) The system (1.1) is elliptic, i.e. for all x in $\bar{G}$, and for all non zero real vector ξ,

$$\alpha_k'(x,\ \xi)\neq 0,\quad k=1,\cdots\cdots,N. \tag{1.5}$$

In the following statements $\tau(x^0)=(\tau_1(x^0),\cdots\cdots,\tau_N(x^0))$ denotes a real non zero vector tangent to ∂G at x^0(on ∂G) and $\nu(x^0)=(\nu_1(x^0),\cdots\cdots,\nu_N(x^0))$ a real non zero vector normal to ∂G at x^0. (For simplicity, we often suppress x^0 and denote these vectors simply by τ, ν respectively).

(II) The system (1.1) is properly elliptic, i. e. for every x^0 on $\partial^{(i)}G$, $i=1,\ 2$, and every pair of vectors $(\tau(x^0),\ \nu(x^0))$, the polynomials (in complex scalar η),

$$a_k'(\eta)=a_k'(x^0;\ \tau+\eta\nu),\quad k=1,\cdots\cdots,N, \tag{1.6}$$

have exactly m roots $\eta_{k,s}^+\ (x^0,\ \tau,\ \nu)$, $s=1,\cdots\cdots,m$, with positive imaginary parts.

Next, we set

$$a_k^+(\eta)=\prod_{s=1}^{m}(\eta-\eta_{k,s}^+(x^0,\ \tau,\ \nu)),\quad k=1,\cdots\cdots,N, \tag{1.7}$$

$$b_{j,k}^{(i)\prime}(\eta)=b_{j,k}^{(i)\prime}(x^0,\ \tau+\eta\nu), \tag{1.8}$$

$$i=1,\ 2;\ j=1,\cdots\cdots,Nm;\ k=1,\cdots\cdots,N.$$

(III) For each x^0 on $\partial^{(i)}G$, $\mathscr{B}^{(i)}(x,\ D)$ complements $\alpha(x,\ D)$, i. e. for every x^0 on $\partial^{(i)}G$, $i=1,2$, and for every pair of vectors $(\tau(x^0),\ \nu(x^0))$, any relation of the form

$$\sum_{j=1}^{Nm}\lambda_j b_{j,k}^{(i)\prime}(\eta)\equiv 0\ (\mathrm{mod}\ a_k^+(\mu)),\quad k=1,\cdots\cdots,N, \tag{1.9}$$

in which the λ_j are complex constants independent of η, imply that $\lambda_j=0$, $j=1,\cdots,Nm$.

Next, let us write,

$$b'_{j,k}(\eta)=\sum_{s=1}^{2m}\beta_s^{(i,j,k)}\eta^{s-1}, \tag{1.10}$$

$$i=1,\ 2;\ j=1,\cdots\cdots,Nm;\ k=1,\cdots\cdots,N.$$

Further, let

$$\pi_k^\sigma(\eta)=\sum_{s=0}^{2m}\rho_s^{(\sigma,k)}\ \eta^{s-1}, \tag{1.11}$$

$$\sigma=1,\cdots\cdots,m;\ k=1,\cdots\cdots,N.$$

Evidently, $\beta_s^{(i,j,k)}$, $\rho_s^{(\sigma,k)}$ depend on $(x^0,\ \tau(x^0),\ \nu(x^0))$.

(IV) For each x^0 on Π and for each vector $\nu(x^0)$, there exist $(Nm)^2$ polynomials in $\tau(x^0)$, namely

$$e_{j,l}(\tau)=\sum_{|\mu|=\nu_j^{(1)}+\nu_l^{(2)}-1}\varepsilon^{(j,l)}(x^0,\ \nu)\tau^\mu,\qquad j,\ l=1,\cdots\cdots,Nm \tag{1.12}$$

such that

$$Re\sum_{j,l=1}^{Nm}e_{j,l}(\tau)\Big(\sum_{k=1}^{N}\sum_{s=1}^{2m}\beta_s^{(1,j,k)}\omega_{s-1}^{(k)}\Big)\Big(\overline{\sum_{k=1}^{N}\sum_{s=1}^{2m}\beta_s^{(2,l,k)}\omega_{s-1}^{(k)}}\Big)>0, \tag{1.13}$$

for all complex vectors $\Omega^{(k)}=(\omega_0^{(k)},\cdots\cdots\cdots,\omega_{2m-1}^{(k)})$, $k=1,\cdots\cdots\cdots,N$, for which

$$\sum_{s=1}^{2m}\rho_s^{(\sigma,k)}\omega_{s-1}^{(k)}=0,\ \ k=1,\cdots\cdots,N;\ \ \sigma=1,\cdots\cdots,m,$$

unless $\Omega^{(k)}=0$, $k=1,\cdots\cdots,N$.

We remark that each $e_{j,l}(\tau)$ is homogeneous in τ of order $\nu_j^{(1)}+\nu_l^{(2)}-1$. Two boundary operators of type (1.2) having the property (IV) are called "compatible with respect to (1.1) on Π".

The complementing condition (III) is a special case of the one introduced by Agmon-Douglis-Nierenberg [3]. The compatibility condition (IV) is a generalization of those of Schechter [8], [10].

2. Preliminary Results. In this section, as the first step, we consider the problem in a hemi-sphere. For convenience, let us change some notations introduced in section 1. We denote a point in E_n by

$$(x,y)=(x_1,\cdots\cdots,x_{n-1},\ y)\ \text{or}\ (\xi,\ \eta)=(\xi_1,\cdots\cdots,\ \xi_{n-1},\ \eta),$$

and correspondingly, we set

$$D_x=\Big(\frac{1}{i}\frac{\partial}{\partial x_1},\cdots\cdots,\frac{1}{i}\frac{\partial}{\partial x_{n-1}}\Big),\ \ D_y=\frac{1}{i}\frac{\partial}{\partial y}.$$

For a multi-index $\mu=(\mu_1,\cdots\cdots\mu_{n-1})$, ξ^μ, D_x^μ are defined as before.

Now, let

$$\Sigma_R=\Big\{(x,\ y)\Big|\ |x|^2+y^2\leqq R^2,\ \ y\geqq 0\Big\},$$

$$\partial_1\Sigma_R=\Big\{(x,\ 0)\Big|\ |x|^2\leqq R^2\Big\},$$

$$\partial_2\Sigma_R=\Big\{(x,\ y)\Big|\ |x|^2+y^2=R^2,\ \ y\geqq 0\Big\}.$$

$\mathscr{F}^N(\Sigma_R)$ stands for the collection of all vector (N-) valued functions of class $C^\infty(\Sigma_R)$ which vanish near $\partial_2\Sigma_R$. (Instead of $\mathscr{F}^1(\Sigma_R)$, we use $\mathscr{F}(\Sigma_R)$.) For a function u of class $\mathscr{F}(\Sigma_R)$, we define

(2.1)
$$|u|_p^2=(|u|_{p,\Sigma_R})^2=\sum_{|\mu|+s=p}\int_{\Sigma_R}|D_x^{\mu}D_y^s u|^2dxdy.$$

If $U=(u_1, \cdots\cdots, u_N)$, $V=(v_1, \cdots\cdots, v_N)$ are both of class $\mathscr{F}^N(\Sigma_R)$, we set

(2.2)
$$U\cdot V=u_1\bar{v}_1+\cdots\cdots\cdots+u_N\bar{v}_N,\quad |U|^2=U\cdot U,$$
$$((U,\ V))=((U,\ V))_{\Sigma_R}=\int_{\Sigma_R}U\cdot V dxdy,$$

(2.3)
$$|U|_p^2=(|U|_{p,\Sigma_R})^2=\sum_{k=1}^{N}|u_k|_p^2,\quad \|U\|_p^2=(\|U\|_{p,\Sigma_R})^2=\sum_{t=0}^{p}|U|_t^2.$$

The difference between $|U|$ and $|U|_0=\|U\|_0$ should be carefully noted.

The basic relation between the norms(2.3) is the following well known lemma.

LEMMA 2.1. *Let u be a function of class $\mathscr{F}(\Sigma_R)$. Then for an arbitrary pair of integers p and t, for which $0\leqq t<p$, and for an arbitrary positive number ε, there exists a positive number $K(\varepsilon)$ independent of u such that*

(2.4)
$$\|u\|_t\leqq\varepsilon\|u\|_p+K(\varepsilon)\|u\|_0.$$

Next, let u be a function of clas $\mathscr{F}(\Sigma_R)$. Extending u as identically zero outside Σ_R, we put

$$\tilde{u}(\xi,\ \eta)=\int_{E_2}e^{-i(\xi x+\eta y)}u(x,\ y)dxdy,\quad \hat{u}(\xi)=\int_{|x|<\infty}e^{-i\xi\cdot x}u(x,\ 0)dx.$$

If U is of class $\mathscr{F}^N(\Sigma_R)$, we define $\tilde{U}=(\tilde{u}_1, \cdots\cdots, \tilde{u}_N)$. If u, v are of class $\mathscr{F}(\Sigma_R)$, and if p is non-negative and real, we define

(2.5)
$$\langle u,\ v\rangle_p=\langle u,\ v\rangle_{p,\partial_1\Sigma_R}=\int_{|\xi|<\infty}|\xi|^{2p}\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi,$$
$$\langle u\rangle_p^2=(\langle u\rangle_{p,\partial_1\Sigma_R})^2=\langle u,\ u\rangle_p.$$

It is clear that (2.5) defines another norm(called boundary norm) on $\mathscr{F}(\Sigma_R)$.

The following lemmas are proved in [7].

LEMMA 2.2. *If u is of class $\mathscr{F}(\Sigma_R)$, we have*

(2.6)
$$\hat{u}(\xi)=-\frac{i}{\pi}\int_{-\infty}^{\infty}\tilde{u}(\xi,\eta)d\eta.$$

LEMMA 2.3. *If u is of class $\mathscr{F}(\Sigma_R)$ and p a non-negative integer, then*

(2.7)
$$\langle u\rangle_{p+\frac{1}{2}}\leqq c\|u\|_{p+1},$$

where c is a positive constant independent of u.

Using Schwarz' inequality, we immediately have the following:

COROLLARY 2.3.1. *If u, v are both of class $\mathscr{F}(\Sigma_R)$ and p is a non-negative* *integer, we have*

$$\langle u, v\rangle_{p+\frac{1}{2}} \leqq c\|u\|_{p+1}\|v\|_{p+1}, \tag{2.8}$$

c being independent of u, v.

Let us now consider the systems of differential operators

$$\mathfrak{A}(D_x, D_y)=\begin{pmatrix} a_1(D_x, D_y) & & 0 \\ & \ddots & \\ 0 & & a_N(D_x, D_y) \end{pmatrix} \tag{2.9}$$

$$a_k(D_x, D_y)=\sum_{|\mu|+s=2m} \alpha_{\mu, s}^{(k)} D_x^{\mu} D_y^{s}, \quad k=1, \cdots\cdots, N,$$

and

$$\mathscr{B}(D_x, D_y)=(b_{j,k}(D_x, D_y)), \tag{2.10}$$

$$b_{j,k}(D_x, D_y)=\sum_{|\mu|+s=2m-\nu_j} \beta_{\mu, s}^{(j,k)} D_x^{\mu} D_y^{s}, \nu_j \geqq 1,$$

$$j=1, \cdots\cdots, Nm; \quad k=1, \cdots\cdots, N,$$

where $\alpha_{\mu, s}^{(k)}$ and $\beta_{\mu, s}^{(j,k)}$ are complex constants. For each ξ, we write the characteristic polynomials of these operators as follows:

$$a_k(\eta)\Bigl(=\sum_{s=0}^{2m}\Bigl(\sum_{|\mu|=2m-s} \alpha_{\mu, s}^{(k)} \xi^{\mu}\Bigr)\eta^{s}\Bigr)=\sum_{s=0}^{2m} \alpha_{s}^{(k)}(\xi)\eta^{s}, \tag{2.11}$$

$$b_{j,k}(\eta)=\sum_{s=1}^{2m} \beta_{s}^{(j,k)}(\xi)\eta^{s-1}, \tag{2.12}$$

$$j=1, \cdots\cdots, Nm; \quad k=1, \cdots\cdots, N.$$

We further assume the following conditions:

I′. For every non zero real vector (ξ, η), $a_k(\xi, \eta)\neq 0$, $(k=1, \cdots\cdots, N)$.

II′. For each non zero real vector ξ, the polynomial $a_k(\eta)=0$ has exactly m roots $\eta_{k,s}^{+}(\xi)$ $(s=1, \cdots\cdots, m)$ with positive imaginary parts $(k=1, \cdots\cdots, N)$.

Set

$$a_k^{+}(\eta)=\prod_{s=1}^{m}(\eta-\eta_{k,s}^{+}(\xi)), \quad k=1, \cdots\cdots, N. \tag{2.13}$$

III′. For each non zero real vector ξ, the relations

$$\sum_{j=1}^{Nm} \lambda_j b_{j,k}(\eta)\equiv 0 \pmod{a_k^{+}(\eta)}, \quad k=1, \cdots\cdots, N$$

with constant λ_j imply $\lambda_j=0$, $j=1, \cdots\cdots, Nm$.

Now we shall prove the following theorem which is also proved in [3] by different methods.

THEOREM 2.1. *If the operators* (2.9), (2.10) *have the properties* I′, II′, III′, *then for all U of class* $\mathscr{F}^N(\Sigma_R)$, *there exists a positive constant c (independent of U) such that*

$$\|U\|^2_{2m}\leqq c\Big(\|\mathscr{A}(D_x, D_y)U\|^2_0+\sum_{j=1}^{Nm}\Big(<\sum_{k=1}^{N} b_{j,k}(D_x, D_y)u_k>_{\nu_j-\frac{1}{2}}\Big)^2+\|U\|^2_0\Big). \tag{2.14}$$

Proof of this theorem depends on Fourier transforms and homogeneity of characteristic polynomials. For later use, we make a few remarks on these.

If $U=(u_1, \cdots\cdots, u_N)$ is of class $\mathscr{F}^N(\Sigma_R)$, we have

$$\tilde{a}_k(D_x, D_y)u_k=a_k(\xi, \eta)\tilde{u}_k+\sum_{s=1}^{2m}\alpha_s^{(k)}(\xi)\sum_{t=1}^{s}\eta^{s-t}\widehat{D_y^{t-1}u_k}.$$

Hence if we set

$$\theta_t^{(k)}\Big(=\theta_t^{(k)}(\xi)\Big)=|\xi|^{-t-1}\widehat{D_y^{t}u_k},\quad t=0, \cdots\cdots, 2m-1, \tag{2.15}$$

we have

$$\tilde{a}_k(D_x, D_y)u_k=a_k(\xi, \eta)\tilde{u}_k+\mathrm{a}_k,\quad k=1, \cdots\cdots, N,$$

where

$$\mathrm{a}_k=\sum_{s=1}^{2m}\alpha_s^{(k)}(\xi)\sum_{t=1}^{s}\eta^{s-t}|\xi|^t\theta_{t-1}^{(k)},\quad k=1, \cdots\cdots, N. \tag{2.16}$$

Let us next set

$$\mathscr{R}_k^{(\sigma)}(\xi, \eta)=|\xi|^{m-\sigma}\eta^{\sigma-1}a_k^+(\xi, \eta)=\sum_{s=1}^{2m}\rho_s^{(\sigma,k)}(\xi)\eta^{s-1}, \tag{2.17}$$
$$k=1, \cdots\cdots, N;\ \sigma=1, \cdots\cdots, m.$$

Then we have

$$\sum_{s=1}^{2m}\rho_s^{(\sigma,k)}(\xi)\widehat{D_y^{s-1}u_k}=\rho_1^{(\sigma,k)}(\xi)\tilde{u}_k+\sum_{s=2}^{2m}\rho_s^{(\sigma,k)}(\xi)\widehat{D_y^{s-1}u_k} \tag{2.18}$$
$$=\mathscr{R}_k^{(\sigma)}(\xi, \eta)\tilde{u}_k+r_k^{(\sigma)},$$
$$k=1, \cdots\cdots, N;\ \sigma=1, \cdots\cdots, m,$$

where

$$r_k^{(\sigma)}=\sum_{s=2}^{2m}\rho_s^{(\sigma,k)}(\xi)\sum_{t=1}^{s-1}\eta^{s-t-1}|\xi|^t\theta_{t-1}^{(k)}, \tag{2.19}$$
$$k=1, \cdots\cdots, N;\ \sigma=1, \cdots\cdots, m.$$

Furthermore from Lemma (2.2) it follows that

$$\int_{-\infty}^{\infty} \sum_{s=1}^{2m} \rho_s^{(\sigma,k)}(\xi)\widetilde{D_y^{s-1}u_k}\, d\eta = -\pi i\Big(\sum_{s=1}^{2m} \rho_s^{(\sigma,k)}(\xi)|\xi|^s\theta_{s-1}^{(k)}\Big). \tag{2.20}$$

We shall furhter make use of the following lemmas. Let us regard

$$q_k^{(\sigma)} = \mathscr{R}_k^{(\sigma)} a_k - a_k r_k^{(\sigma)}, \quad k=1,\cdots\cdots,N;\ \sigma=1,\cdots\cdots,m.$$

as polynomials in η. Then we have

LEMMA 2.4. *deg* $q_k^{(\sigma)} \leqq 2m-1$ (*in* η) *and*

$$\int_{-\infty}^{\infty} \frac{q_k^{(\sigma)}}{a_k} d\eta = -\pi i\Big(\sum_{s=1}^{2m} \rho_s^{(\sigma,k)}(\xi)|\xi|^s\theta_{s-1}^{(k)}\Big) \tag{2.21}$$

$$k=1,\cdots\cdots,N;\ \sigma=1,\cdots\cdots,m.$$

where the integral is taken over the real axis in the Cauchy principal value sense.

Next, let ξ be a fixed, real, non-zero vector.

If $\Omega^{(k)} = (\omega_0^{(k)},\cdots\cdots,\omega_{2m-1}^{(k)})$, $k=1,\cdots\cdots,N$, are complex vectors, we have

LEMMA 2.5. *In order that* $\sum_{j=1}^{Nm}\lambda_j b_{j,k}(\eta) \equiv 0 \pmod{a_k^+(\eta)}$, $k=1,\cdots\cdots,N$,

shall imply $\lambda_j=0$, $j=1,\cdots\cdots,Nm$, *it is necessary and sufficient that*

$$\sum_{k=1}^{N}\sum_{s=1}^{2m} \beta_s^{(j,k)}(\xi)\omega_{s-1}^{(k)}=0,\quad j=1,\cdots\cdots,Nm,$$

$$\sum_{s=1}^{2m} \rho_s^{(\sigma,k)}(\xi)\omega_{s-1}^{(k)}=0,\quad k=1,\cdots\cdots,N;\ \sigma=1,\cdots\cdots m,$$

imply $\Omega^{(k)}=0$, $k=1,\cdots\cdots,N$.

LEMMA 2.6. *If* $h(\xi,\eta)$ *is a function which is homogeneous of order* t *in* (ξ,η), *then*

$$g(\xi) = \int_{-\infty}^{\infty} h(\xi,\eta)d\eta$$

is homogeneous of order $t+1$ *in* ξ *whenever the integral in the right side exists.*

The above three lemmas will be proved in the Appendix.

Let $\Omega=(\omega_1,\cdots\cdots,\omega_p)$, $\Lambda=(\lambda_1,\cdots\cdots,\lambda_q)$, be two variable vectors. Then we have the following lemma which is of trivial nature.

LEMMA 2.7. *Let* $h(\Omega,\Lambda)$, $g(\Omega,\Lambda)$ *be continuous functions of* Ω, Λ *which are both homogeneous of order* s *with respect to* Ω, *and of order* t *with respect to* Λ. *If* $h(\Omega,\Lambda)\neq 0$ *for* $\Omega\neq 0$, $\Lambda\neq 0$, *then there exists a positive number* c *such that*

$$|g(\Omega,\Lambda)| \leqq c|h(\Omega,\Lambda)|$$

for all Ω and Λ.

In the following computations, we shall ordinarily omit writing the arguments ξ, η of characteristic polynomials. The operators will be denoted with D_x, D_y.

Proof of Theorem 2.1. Let U be of class $\mathscr{F}^N(\Sigma_R)$ and, for any multi-index μ and positive integer s such that $|\mu|+s=2m$, let

$$\mathscr{G}(D_x, D_y)=\begin{pmatrix} D_x^\mu D_y^s & & 0 \\ & \ddots & \\ 0 & & D_x^\mu D_y^s \end{pmatrix}. \tag{2.22}$$

Our aim is to show that

$$\int|\tilde{\mathfrak{A}}(D_x, D_y)U|^2 d\eta+\sum_{j=1}^{Nm}|\xi|^{2\nu_j-1}\Big|\sum_{k=1}^{N}\hat{b}_{j,k}(D_x, D_y)u_k\Big|^2-c_0\int|\tilde{\mathscr{G}}(D_x, D_y)|^2 d\eta \geqq 0, \tag{2.23}$$

with some positive constant c_0 which is independent of U. The desired result would be obtained if we summed up (2.23) over all possible choices of $\mathscr{G}(D_x, D_y)$ for which $|\mu|+s=2m$ and integrate again with respect to ξ. In fact, invoking Parceval's relation, we obtain

$$\|\mathfrak{A}(D_x, D_y)U\|_0^2+\sum_{j=1}^{Nm}<\sum_{k=1}^{N}b_{j,k}(D_x, D_y)u_k>_{\nu_j-\frac{1}{2}}\geqq c|U|_{2m}^2.$$

Hence applying Lemma 2.1, we would have (2.14). To prove (2.23), we first consider the following. Let

$$\Lambda^{(k)}=\Lambda^{(k)}(\xi)=(\lambda_1^{(k)}(\xi), \cdots\cdots, \lambda_m^{(k)}(\xi)),\quad k=1, \cdots\cdots, N,$$

be a function of ξ to be chosen later. Then we have

$$\begin{aligned}\Big|\tilde{\mathfrak{A}}(D_x, D_y)U\Big|^2&=\sum_{k=1}^{N}|a_k\tilde{u}_k+\mathrm{a}_k|^2\\&=\sum_{k=1}^{N}\Big\{\Big|a_k\tilde{u}_k+\mathrm{a}_k-\frac{1}{\bar{a}_k}\sum_{\sigma=1}^{m}\overline{\lambda_\sigma^{(k)}}\,\overline{\mathscr{R}_k^{(\sigma)}}\Big|^2+2Re\sum_{\sigma=1}^{m}\overline{\lambda_\sigma^{(k)}}\Big(\sum_{s=1}^{2m}\rho_s^{(\sigma,k)}(\xi)\widetilde{D_y^{s-1}u_k}\Big)\\&\qquad+2Re\sum_{\sigma=1}^{m}\frac{\overline{\lambda_\sigma^{(k)}}}{a_k}(\mathscr{R}_k^{(\sigma)}\mathrm{a}_k-a_k r_k^{(\sigma)})-\Big|\frac{1}{a_k}\sum_{\sigma=1}^{m}\lambda_\sigma^{(k)}\overline{\mathscr{R}_k^{(\sigma)}}\Big|^2\Big\}\end{aligned}$$

where we made use of the relation (2.18). Noting that for each fixed ξ, $\eta^{(\sigma)}\mathrm{a}_k-a_k r_k^{(\sigma)}$ is the expression denoted by $q_k^{(\sigma)}$ in Lemma 2.4, we have

$$\int\frac{1}{a_k}(\mathscr{R}_k^{(\sigma)}\mathrm{a}_k-a_k r_k^{(\sigma)})d\eta=-\pi i\Big(\sum_{s=1}^{2m}\rho_s^{(\sigma,k)}(\xi)|\xi|^s\theta_{s-1}^{(k)}\Big),$$

$$k=1, \cdots\cdots, N;\ \sigma=1, \cdots\cdots, m.$$

Hence, using (2.20), we obtain

$$\int \left| \widetilde{\alpha}(D_x, D_y)U \right|^2 d\eta = \sum_{k=1}^{N} \left\{ \int \left| a_k \tilde{u}_k + a_k - \frac{1}{\bar{a}_k} \sum_{\sigma=1}^{m} \lambda_\sigma^{(k)} \overline{\mathscr{R}}_k^{(\sigma)} \right|^2 d\eta \right.$$
$$\left. + 4Re(-\pi i) \sum_{\sigma=1}^{m} \overline{\lambda}_\sigma^{(k)} \left(\sum_{s=1}^{2m} \rho_s^{(\sigma,k)}(\xi) |\xi|^s \theta_{s-1}^{(k)} \right) - \int \left| \frac{1}{\bar{a}_k} \sum_{\sigma=1}^{m} \lambda_\sigma^{(k)} \overline{\mathscr{R}}_k^{(\sigma)} \right|^2 d\eta \right\}$$

Now let us pick

$$\lambda_\sigma^{(k)}(\xi) = -2\pi i \varepsilon |\xi| \left(\sum_{s=1}^{2m} \rho_s^{(\sigma,k)}(\xi) |\xi|^s \theta_{s-1}^{(k)} \right), \quad k=1, \cdots\cdots, N; \quad \sigma=1, \cdots\cdots, m,$$

ε being a positive constant to be chosen later. We then have

$$\int |\widetilde{\alpha}(D_x, D_y)U|^2 d\eta = \sum_{k=1}^{N} \left\{ \int \left| a_k \tilde{u}_k + m_k \right|^2 d\eta + \frac{2}{\varepsilon |\xi|} \sum_{\sigma=1}^{m} \left| \lambda_\sigma^{(k)} \right|^2 - \int \left| \frac{1}{\bar{a}_k} \sum_{\sigma=1}^{m} \overline{\mathscr{R}}_k^{(\sigma)} \lambda_\sigma^{(k)} \right|^2 d\eta \right\}, \tag{2.24}$$

where we have set

$$m_k = a_k - \frac{1}{\bar{a}_k} \sum_{\sigma=1}^{m} \lambda_\sigma^{(k)} \overline{{}_k\mathscr{R}}^{(\sigma)}, \qquad k=1, \cdots\cdots, N,$$

Now for each k, the last two terms in (2.24) are quadratic in $A^{(k)}$. Furthermore, from Lemma 2.6 it follows that each coefficient of $\lambda_\sigma^{(k)} \overline{\lambda}_\iota^{(k)}$ is continuous and homogeneous of order -1 in ξ. Hence, by Lemma 2.7, we find a positive constant ε such that

$$\int \left| \frac{1}{\bar{a}_k} \sum_{\sigma=1}^{m} \overline{\mathscr{R}}_k^{(\sigma)} \lambda_\sigma^{(k)} \right|^2 d\eta \leqq \frac{1}{\varepsilon |\xi|} \sum_{\sigma=1}^{m} \left| \overline{\lambda_\sigma^{(k)}} \right|^2, \quad k=1, \cdots\cdots, N.$$

Inserting this value of ε in (2.24), we have

$$\int |\widetilde{\alpha}(D_x, D_y)U|^2 d\eta \geqq \sum_{k=1}^{N} \left\{ \int \left| a_k \tilde{u}_k + m_k \right|^2 d\eta + \frac{1}{\varepsilon |\xi|} \sum_{\sigma=1}^{m} \left| \lambda_\sigma^{(k)} \right|^2 \right\} \tag{2.25}$$

This is the first step of our estimate. Next let us consider the sum of the first two terms of (2.23). Since

$$\widehat{b_{j,k}(D_x, D_y) u_k} = \sum_{|\mu|+s=2m-\nu_j} \beta_{\mu,s}^{(j,k)} \xi^\mu \widehat{D_y^s u_k} = \sum_{s=1}^{2m} \beta_s^{(j,k)}(\xi) |\xi|^s \theta_{s-1}^{(k)}$$

we have

$$\begin{aligned} &\int \left| \widetilde{\alpha}(D_x, D_y)U \right|^2 d\eta + \sum_{j=1}^{Nm} |\xi|^{2\nu_j - 1} \left| \sum_{k=1}^{N} \widehat{b}_{j,k}(D_x, D_y) u_k \right|^2 \\ &\geqq \sum_{k=1}^{N} \int \left| a_k \tilde{u}_k + m_k \right|^2 d\eta + \frac{1}{\varepsilon |\xi|} \sum_{k=1}^{N} \sum_{\sigma=1}^{m} \left| \lambda_\sigma^{(k)} \right|^2 \\ &+ \sum_{j=1}^{Nm} |\xi|^{2\nu_j - 1} \left| \sum_{k=1}^{N} \sum_{s=1}^{2m} \beta_s^{(j,k)}(\xi) |\xi|^s \theta_{s-1}^{(k)} \right|^2 \end{aligned} \tag{2.26}$$

The last two terms in the right side of (2.26) are quadratic in $\Theta^{(k)}$; each coefficient of $\theta_s^{(k)}\bar{\theta}_t^{(k)}$ is continuous and homogeneous of order $4m+1$ in ξ. Hence we can express it in the form

$$(2.27)\qquad \sum_{k=1}^{N}\sum_{s,t=0}^{2m-1} H_{s,t}^{(k)}\theta_s^{(k)}\bar{\theta}_t^{(k)},$$

where each $H_{s,t}^{(k)}(\xi)$ is continuous and homogeneous of order $4m+1$ in ξ. Furthermore, from condition III′ and Lemma 2.5, it follows that

$$(2.28)\qquad \sum_{k=1}^{N}\sum_{s,t=0}^{2m-1} H_{s,t}^{(k)}\theta_s^{(k)}\bar{\theta}_k^{(k)}>0$$

unless $\Theta^{(k)}=0$, $k=1,\cdots\cdots,N$. Indeed, if (2.28) were zero, each summand in the last two terms of (2.26) would vanish. However, the relations $\lambda_\sigma^{(k)}=0$, $k=1,\cdots\cdots,N$; $\sigma=1,\cdots\cdots,m$, are equivalent to $\sum_{s=1}^{2m}\rho_s^{(\sigma,k)}(\xi)\omega_{s-1}^{(k)}=0$, $k=1,\cdots\cdots\cdots,N$; $\sigma=1,\cdots\cdots,m$ where we replaced $|\xi|^s\theta_{s-1}^{(k)}$ by $\omega_{s-1}^{(k)}$.

On the other hand, the vanishing of the last term of (2.26) implies that $\sum_{k=1}^{N}\sum_{s=0}^{2m}\beta_s^{(j,k)}(\xi)\omega_{s-1}^{(k)}=0$, $j=1,\cdots\cdots,Nm$.

Hence from Lemma 2.5 we have $\Theta^{(k)}=0$, $k=1,\cdots\cdots,N$.

Finally, we proceed to complete the proof. Let c_0 be a positive constant to be choosen later. Then

$$(2.29)\qquad \int\Big|\widetilde{\mathcal{A}}(D_x,D_y)U\Big|^2+\sum_{j=1}^{Nm}|\xi|^{2\nu_j-1}\Big|\sum_{k=1}^{N}\hat{b}_{j,k}(D_x,D_y)u_k\Big|^2-c_0\int\Big|\widetilde{\mathcal{G}}(D_x,D_y)U\Big|^2d\eta$$
$$\geqq\sum_{k=1}^{N}\int\Big|a_k\tilde{u}_k+m_k\Big|^2d\eta+\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}H_{s,t}^{(k)}\theta_s^{(k)}\bar{\theta}_t^{(k)}-c_0\sum_{k=1}^{N}\int\Big|g_k\tilde{u}_k+g_k\Big|^2d\eta,$$

where in the last term, we set

$$g_k=\xi^\mu\eta^s,\ (|\mu|+s=2m),\quad g_k=\xi^\mu\sum_{t=1}^{s}\eta^{s-t}|\xi|^t\theta_{t-1}^{(k)},\ k=1,\cdots\cdots,N.$$

Setting $f_k=|a_k|^2-c_0|g_k|^2$, $k=1,\cdots\cdots,N$ we rewrite the right side of (2.29) as follows:

$$(2.30)\qquad \sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}H_{s,t}^{(k)}\theta_s^{(k)}\bar{\theta}_t^{(k)}+\sum_{k=1}^{N}\int\Big\{(|a_k|^2-c_0|g_k|^2)|\tilde{u}_k|^2+|m_k|^2$$
$$-c_0|g_k|^2+2Re(\bar{a}_km_k-c_0\bar{g}_kg_k)\tilde{u}_k\Big\}d\eta=\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}H_{s,t}^{(k)}\theta_s^{(k)}\bar{\theta}_t^{(k)}$$
$$+\sum_{k=1}^{N}\int\Big\{f_k\Big|\tilde{u}_k+\frac{1}{f_k}(\bar{a}_km_k-c_0\bar{g}_kg_k)\Big|^2-\frac{c_0}{f_k}\Big|a_kg_k-g_km_k\Big|^2\Big\}d\eta$$

$$=\sum_{k=1}^{m}\sum_{s,t=0}^{2m-1}H_{s,t}^{(k)}\theta_s^{(k)}\overline{\theta}_t^{(k)}+\sum_{k=1}^{N}\int f_k\left|\tilde{u}^k+\frac{1}{f_k}(\bar{a}_k m_k-c_0\bar{g}_k g_k)\right|^2 d\eta$$

$$-c_0\sum_{k=1}^{N}\int\frac{1}{f_k}\left|a_k g_k-g_k m_k\right|^2 d\eta.$$

Now, using lemma 2.6, we can express the last term of the right side of (2.30) in the form

$$\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}L_{s,t}^{(k)}\theta_s^{(k)}\overline{\theta}_t^{(k)}, \tag{2.31}$$

where each $L_{s,t}^{(k)}$ is continuous and homogeneous of order $4m+1$ in ξ. However, since $\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}H_{s,t}^{(k)}\overline{\theta}_t^{(k)}$ has the same property and is positive definite, there exists a positive number ε_2 such that

$$\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}H_{s,t}\theta_s^{(k)}\overline{\theta}_t^{(k)}\geqq 2c_0\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}L_{s,t}^{(k)}\theta_s^{(k)}\overline{\theta}_t^{(k)}, \tag{2.32}$$

if $0\leqq c_0\leqq\varepsilon_2$. Therefore, if we take c_0 so that $0<c_0<\min(\varepsilon_1,\varepsilon_2)$ in (2.29), from (2.29), (2.30) and (2.32), it follows that

$$\int|\widetilde{\mathcal{A}}(D_x,D_y)U|^2d\eta+\sum_{j=1}^{Nm}|\xi|^{2\nu_j-1}\left|\sum_{k=1}^{N}\widehat{b}_{j,k}(D_x,D_y)u_k\right|^2-c_0\int|\widetilde{\mathcal{G}}(D_x,D_y)U|^2d\eta$$

$$\geqq\frac{1}{2}\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}H_{s,t}^{(k)}\theta_s^{(k)}\overline{\theta}_t^{(k)}+\sum_{k=1}^{N}\int f_k\left|\tilde{u}_k+\frac{1}{f_k}(\bar{a}_k m_k-c_0\bar{g}_k g_k)\right|^2d\eta>0.$$

Thus the proof is complete.

3. Integral estimates. In this section, we consider the system (2.9), together with a mixed boundary system.

Let Π be an n-2 dimensional manifold on $\partial_1\Sigma_R$ which passes through the origin and divides $\partial_1\Sigma_R$ into two components $\partial_1^{(1)}\Sigma_R$ and $\partial_1^{(2)}\Sigma_R$.

By $\mathcal{F}^N(\Sigma_R,\Pi)$, we denote the functions of class $\mathcal{F}^N(\Sigma_R)$ which vanish near Π. Now for a function u of class $\mathcal{F}(\Sigma_R,\Pi)$, we put

$$u^{(i)}(x)=\begin{cases}u(x,\ 0), & x\in\partial_1^{(i)}\Sigma_R\\ 0, & x\notin\partial_1^{(i)}\Sigma_R, \qquad i=1,2.\end{cases}$$

If u, v are both of class $\mathcal{F}(\Sigma_R,\Pi)$, we define

$$<u,\ v>_p^{(i)}=<u,v>_{p,\partial_1^{(i)}\Sigma_R}=\int_{|\xi|<\infty}|\xi|^{2p}\widehat{u}^{(i)}\bar{\hat{v}}^{(i)}d\xi,$$

$$(<u>_p^{(i)})^2=(<u>_{p,\partial_1^{(i)}\Sigma_R})^2=<u,\ u>_p^{(i)},\quad i=1,\ 2.$$

Next, let us make the following observation.
If $h(\xi)$ is a polynomial in ξ, and if u, v are functions of class $\mathscr{F}(\Sigma_R, \Pi)$, we have

$$\int_{|\xi|} h(\xi)\hat{u}^{(1)}(\xi)\bar{\hat{v}}^{(2)}d\xi=\int_{|x|<\infty}(h(D_x)u^{(1)}(x))v^{(2)}(x)dx=0$$

and a similar relation satisfied by $\hat{v}^{(1)}$ and $\hat{u}^{(2)}$.
Consequently,

$$\int_{|\xi|<\infty} h(\xi)\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi=\int_{|\xi|<\infty} h(\xi)\left\{\hat{u}^{(1)}(\xi)\bar{\hat{v}}^{(1)}(\xi)+\hat{u}^{(2)}(\xi)\bar{\hat{v}}^{(2)}(\xi)\right\}d\xi.$$

If, in particular, $h(\xi)$ is a homogeneous polynomial of order $t+s$ in ξ, we have

$$\left|\int h(\xi)\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi\right|^2\leqq 2\int|h(\xi)|^{\frac{2s}{s+t}}|\hat{u}^{(1)}(\xi)|^2d\xi\int|h(\xi)|^{\frac{2t}{s+t}}|\hat{v}^{(1)}(\xi)|^2d\xi$$
$$+2\int|h(\xi)|^{\frac{2s}{s+t}}|\hat{u}^{(2)}(\xi)|^2d\xi\int|h(\xi)|^{\frac{2t}{s+t}}|\hat{v}^{(2)}(\xi)|^2d\xi.$$

Since $|h(\xi)|^{\frac{2s}{s+t}}$, $|h(\xi)|^{\frac{2t}{s+t}}$ are homogeneous of order $2s$ and $2t$ respectively, there exists a positive constant c such that

$$\sqrt{2}\,|h(\xi)|^{\frac{2s}{s+t}}\leqq c|\xi|^{2s},\quad \sqrt{2}\,|h(\xi)|^{\frac{2t}{s+t}}\leqq c|\xi|^{2t}.$$

Therefore, we have

$$\left|\int h(\xi)\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi\right|\leqq c\left\{\langle u\rangle_s^{(1)}\langle v\rangle_t^{(1)}+\langle u\rangle_s^{(2)}\langle v\rangle_t^{(2)}\right\}.$$

Using the trivial inequality $|ab|\leqq\varepsilon|a|^2+\frac{1}{\varepsilon}|b|^2$, we finally obtain

$$\left|\int h(\xi)\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi\right|\leqq c\left\{\varepsilon(\langle u\rangle_s^{(2)})^2+\varepsilon(\langle v\rangle_t^{(1)})^2+\frac{1}{\varepsilon}(\langle v\rangle_t^{(2)})^2\right\}\tag{3.1}$$
$$+\frac{1}{\varepsilon}(\langle v\rangle_t^{(2)})^2,$$

where ε is an arbitrary positive constant, and c is independent of u, v. We also remark that if p is a non negative integer, then:

$$\langle u\rangle_{p+\frac{1}{2}}^{(i)}\leqq c\|u\|_{p+1},\tag{3.2}$$
$$|\langle u,v\rangle_{p+\frac{1}{2}}^{(i)}|\leqq c\|u\|_{p+1}\|v\|_{p+1},\quad i=1,\ 2,$$

c being independent of u, v.

Now let $\mathfrak{a}(D_x, D_y)$ be the operator (2.9). Let further

$$(3.3)\qquad \mathscr{B}^{(i)}(D_x, D_y)=\left((b_{j,k}{}^{(i)}(D_x, D_y)\right),$$
$$b_{j,k}{}^{(i)}(D_x, D_y)=\sum_{|\mu|+s=2m-\nu_j{}^{(i)}}\beta_{\nu,s}{}^{(i,j,k)}D_x{}^{\mu}D_y{}^{s},\quad \nu_j{}^{(i)}\geqq 1,$$
$$i=1,\ 2;\ j=1,\cdots\cdots,Nm;\ k=1,\cdots\cdots,N,$$

the $\beta_{\nu,s}{}^{(i,j,k)}$ being complex constants. As before we shall express the characteristic polynomials in the form:

$$(3.4)\qquad b_{j,k}{}^{(i)}(\xi,\eta)=\sum_{s=1}^{2m}\beta_s{}'^{(i,j,k)}(\xi)\eta^{s-1}.$$

In addition to the conditions I′, II′, we further assume the following condition which corresponds to IV of section 1.

IV′. There exist $(Nm)^2$ polynomials

$$e_{j,l}(\xi),\quad j,l=1,\cdots\cdots,Nm,$$

which are homogeneous of order $\nu_j{}^{(1)}+\nu_l{}^{(2)}-1$ in ξ such that, if ξ is a non-zero real vector, then

$$(3.5)\qquad Re\sum_{j,l=1}^{Nm}e_{j,l}(\xi)\Big(\sum_{k=1}^{N}\sum_{s=1}^{2m}\beta_s{}^{(1,j,k)}(\xi)\omega_{s-1}{}^{(k)}\Big)\overline{\Big(\sum_{k=1}^{N}\sum_{s=1}^{2m}\beta_s{}^{(2,l,k)}(\xi)\omega_{s-1}{}^{(k)}\Big)}>0$$

for all complex vectors $\Omega^{(k)}=(\omega_0{}^{(k)},\cdots\cdots,\omega_{2m-1}{}^{(k)})$, $k=1,\cdots\cdots,N$, which satisfy

$$(3.6)\qquad \sum_{s=1}^{2m}\rho_s{}^{(\sigma,k)}(\xi)\omega_{s-1}=0,\quad \sigma=1,\cdots\cdots,m;\ k=1,\cdots\cdots,N,$$

unless $\Omega^{(k)}=0$, $k=1,\cdots\cdots,N$.

We recall that $\rho_s{}^{(\sigma,k)}(\xi)$ are the coefficients of $\mathscr{R}_k{}^{(\sigma)}(\xi,\eta)$[c. f. (2.17)]. We shall prove the following:

THEOREM 3. *If the operators* (2.9), (3.3) *satisfy the conditions* I′, II′, IV′, *then for all U of class $\mathscr{F}^N(\Sigma_R,\ \Pi)$, there exists a positive constant c (independent of U) such that*

$$(3.7)\qquad \|U\|_{2m}{}^2\leqq c\Big\{\|\mathfrak{A}(D_x, D_y)U\|_0^2+\sum_{j=1}^{Nm}\Big(<\sum_{k=1}^{N}b_{j,k}{}^{(1)}(D_x, D_y)u_k>_{\nu_j^{(1)}-\frac{1}{2}}^{(2)}\Big)^2$$
$$+\sum_{l=1}^{Nm}\Big(<\sum_{k=1}^{N}b_{l,k}{}^{(2)}(D_x, D_y)u_k>_{\nu_l^{(2)}-\frac{1}{2}}^{(2)}+\|U\|_0^2\Big\}$$

PROOF. We shall show that for all U of class $\mathscr{F}^N(\Sigma_R,\ \Pi)$, the following inequality holds.

$$(3.8)\qquad \int\big|\widetilde{\mathfrak{A}}(D_x, D_y)U\big|^2d\eta+Re\delta\sum_{j,l=1}^{Nm}e_{j,l}(\xi)\sum_{k=1}^{N}\widehat{b}_{j,k}{}^{(1)}(D_x, D_y)u_k\overline{\sum_{k=1}^{N}\widehat{b}_{l,k}{}^{(2)}(D_x, D_y)u_k}$$
$$-c_0\int|\widetilde{\mathscr{G}}(D_x, D_y)U|^2d\eta\geqq 0$$

δ, c_0 being positive constants to be choosen later and $\mathscr{G}(D_x, D_y)$ being defined by

(2.22). If we integrate (3.8) with respect to ξ, using Parceval's relation and (3.1), we have

$$\|\mathfrak{A}(D_x, D_y)U\|_0^2+c_1\delta\sum_{j,l=1}^{Nm}\Big\{\varepsilon\Big(<\sum_{k=1}^{N} b_{j,k}^{(1)}(D_x, D_y)u_k>^{(2)}_{\nu_j^{(1)}-\frac{1}{2}}\Big)^2$$
$$+\varepsilon\Big(<\sum_{k=1}^{N} b_{l,k}^{(2)}(D_x, D_y)u_k>^{(1)}_{\nu_l^{(1)}-\frac{1}{2}}\Big)^2+\frac{1}{\varepsilon}\Big(<\sum_{k=1}^{N} b_{j,k}^{(1)}(D_x, D_y)u_k>^{(1)}_{\nu_j^{(1)}-\frac{1}{2}}\Big)^2$$
$$+\frac{1}{\varepsilon}\Big(<\sum_{k=1}^{N} b_{l,k}^{(2)}(D_x, D_y)u_k>^{(2)}_{\nu_l^{(1)}-\frac{1}{2}}\Big)^2\Big\}-c_2|U|_{2m}\geqq 0.$$

Taking ε sufficiently small and applying (3.2) and Lemma 2.1, we obtain the desired result.

Now let us set

$$J=Re\sum_{j,l=1}^{Nm} e_{j,l}(\xi)\sum_{k=1}^{N}\widehat{b}_{j,k}^{(1)}(D_x, D_y)u_k\sum_{k=1}^{N}\widehat{b}_{l,k}^{(2)}(D_x, D_y)u_k$$
$$=Re\sum_{j,l=1}^{Nm} e_{j,l}(\xi)\Big(\sum_{k=1}^{N}\sum_{s=1}^{2m}\beta_s^{(1,j,k)}(\xi)|\xi|^s\theta_{s-1}^{(k)}\Big)\overline{\Big(\sum_{k=1}^{N}\sum_{s=1}^{2m}\beta_s^{(2,l,k)}(\xi)|\xi|^s\theta_{s-1}^{(k)}\Big)}$$

Then from (2.25), it follows that

$$\int|\widetilde{\mathfrak{A}}(D_x, D_y)U|^2d\eta+Re\delta\sum_{j,l=1}^{Nm}e_{j,l}(\xi)\sum_{k=1}^{N}\widehat{b}_{j,k}^{(1)}(D_x, D_y)u_k\sum_{k=1}^{N}\overline{\widehat{b}_{l,k}^{(2)}(D_x, D_y)u_k} \tag{3.9}$$
$$\geqq\sum_{k=1}^{N}\int|a_k\widetilde{u}_k+m_k|^2d\eta+\frac{1}{\varepsilon|\xi|}\sum_{k=1}^{N}\sum_{\sigma=1}^{m}|\lambda_\sigma^{(k)}|^2+\delta J.$$

Since J is a quadratic form in $\Theta^{(k)}$, $k=1,\cdots\cdots,N$, and each coefficient of $\theta_s^{(k)}\overline{\theta}_t^{(k)}$ is continuous and homogeneous of order $4m+1$ in ξ, the sum of the last two terms in the right side of (3.9) is expressed in the form

$$\frac{1}{\varepsilon|\xi|}\sum_{k=1}^{N}\sum_{\sigma=1}^{m}|\lambda_\sigma^{(k)}|^2+\delta J=\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}K_{s,t}^{(k)}\theta_s^{(k)}\overline{\theta}_t^{(k)}, \tag{3.10}$$

where each $K_{s,t}^{(k)}$ is continuous and homogeneous of order $4m+1$ in ξ. Consequently, we have

$$\int|\widetilde{\mathfrak{A}}(D_x, D_y)U|^2d\eta+Re\delta\sum_{j,l=1}^{Nm}e_{j,l}(\xi)\sum_{k=1}^{N}\widehat{b}_{j,k}^{(1)}(D_x, D_y)u_k\sum_{k=1}^{N}\overline{\widehat{b}_{l,k}^{(2)}(D_x, D_y)u_k}$$
$$-c_0\int|\widetilde{\mathscr{G}}(D_x, D_y)U|^2d\eta$$
$$\geqq\sum_{k=1}^{N}\int|a_k\widetilde{u}_k+m_k|^2d\eta+\sum_{k=1}^{N}\sum_{s,t=0}^{2m-1}K_{s,t}^{(k)}\theta_s^{(k)}\overline{\theta}_t^{(k)}-c_0\sum_{k=1}^{N}\int|g_k\widetilde{u}_k+g_k|^2d\eta,$$

which correspond to the inequality (2.29). Hence, if we show that(3.10) is positive definite, (3.8) follows in exactly the same fashion as the last part of the proof for Theorem 2.1.

Now if $\xi\neq 0$, $\Theta^{(k)}\neq 0$, $k=1,\cdots\cdots,N$, the sign of the range of (3.10) is determined by the sign on $\mathfrak{T}$, the set of points $(\xi, \Theta^{(1)},\cdots\cdots,\Theta^{(N)})$ in $4mN+n$-1 dimensional Euclidean space (ξ being real, $\Theta^{(k)}$ being complex vectors) for which

$$|\xi|^2=|\Theta^{(1)}|^2+\cdots\cdots+|\Theta^{(N)}|^2=1.$$

Let $\mathfrak{T}'$ be the subset of $\mathfrak{T}$ on which

$$\sum_{k=1}^{N}\sum_{\sigma=1}^{m}|\lambda_\sigma^{(k)}|^2=0$$

i.e.

$$\sum_{k=1}^{2m}\rho_s^{(\sigma,k)}(\xi)\omega_{s-1}^{(k)}=0,\quad k=1,\cdots\cdots,N;\ \sigma=1,\cdots\cdots,m,$$

where we replace $|\xi|^s\theta_{s-1}^{(k)}$ by $\omega_{s-1}^{(k)}$.

Since $J>0$ on $\mathfrak{T}'$ by condition IV′, there exists an open set $\mathfrak{P}$ containing $\mathfrak{T}'$ on which $J>0$. On the other hand, on the compact set $\mathfrak{T}-\mathfrak{P}$, $\sum_{k=1}^{N}\sum_{\sigma=1}^{m}|\lambda_\sigma^{(k)}|^2>0$. Hence we find a positive constant δ such that

$$\frac{1}{2\varepsilon|\xi|}\sum_{k=1}^{N}\sum_{\sigma=1}^{m}|\lambda_\sigma^{(k)}|^2\geqq\delta J$$

on $\mathfrak{T}-\mathfrak{P}$. Inserting this value of δ in (3.8), we have

$$\frac{1}{\varepsilon|\xi|}\sum_{k=1}^{N}\sum_{\sigma=1}^{m}|\lambda_\sigma^{(k)}|^2+\delta J\geqq\frac{1}{2\varepsilon|\xi|}\sum_{k=1}^{N}\sum_{\sigma=1}^{m}|\lambda_\sigma^{(k)}|^2>0$$

on $\mathfrak{T}-\mathfrak{P}$. Thus (3.10) is strictly positive and therefore the proof is complete.

Now, we are in a position to deal with the original problem (1.1), (1.2). Using standard methods of partition of the domain $\bar{G}$ into appropriate subdomains and with the suitable modifications of the norms concerned, it follows from Theorems 2.1 and 3.1 that:

THEOREM 3.2. *If the operators* (1.1) *and* (1.2) *satisfy the conditions I, II, III, IV of section* 1, *there exists a positive constant c (independent of U) such that*

$$\|U\|_{2m}^2\leqq c\left(\|\mathfrak{A}(x,\ D)U\|_0^2+\sum_{i=1}^{2}\sum_{j=1}^{Nm}\left(\left\langle\sum_{k=1}^{N}b_{j,k}^{(i)}(x,D)u_k\right\rangle^{(i)}_{\nu_j^{(i)}-\frac{1}{2}}\right)^2+\|U\|_0^2\right)\tag{3.11}$$

for all U of class $\varepsilon^N(\bar{G},\Pi)$, *where* $\varepsilon^N(G,\Pi)$ *denotes the class of vector*

(N-) valued functions of class C^∞ which vanish near Π.

4. Appendix. Here we shall prove Lemma 2.4, 2.5 and 2.6. For this purpose, we prepare the following:

LEMMA. *Let $\mathfrak{a}(\eta)$ be a polynomial (in a complex variable η) of order t with leading coefficient α and having no real roots. Let further $q(\eta)$ be a polynomial of order t-1 which has all roots of $\mathfrak{a}(\eta)$ (and with the same multiplicities) which lie above the real axis. If β is the leading coefficient of $q(\eta)$, we have*

$$\int_{-\beta}^{\beta}\frac{q(\eta)}{\mathfrak{a}(\eta)}d\eta=-\pi i\frac{\beta}{\alpha}$$

where the integral is taken over the real axis in the Cauchy principal value sense.

PROOF. With $R>0$, let Σ_R, $\partial\Sigma_R$, $\partial_1\Sigma_R$, $\partial_2\Sigma_R$, be as in Section 2. By assumption, we have

$$\int_{\partial\Sigma_R}\frac{q(\eta)}{\mathfrak{a}(\eta)}d\eta=\int_{\partial_1\Sigma_R}\frac{q(\eta)}{\mathfrak{a}(\eta)}d\eta+\int_{\partial_2\Sigma_R}\frac{q(\eta)}{\mathfrak{a}(\eta)}d\eta=0.$$

On the other hand,

$$\int_{\partial_2\Sigma_R}\frac{q(\eta)}{\mathfrak{a}(\eta)}d\eta=\pi i\frac{\beta}{\alpha}+O\Big(\frac{1}{R}\Big).$$

Letting $R\to\infty$, we have the desired result.

PROOF of LEMMA 2.4. Since the proof of Lemma 2.4 is the same for each k, we shall suppress the index k. Now for each fixed ξ, we have, from (2.11), (2.16), (2.17) and (2.19),

$$\begin{aligned}
&\mathfrak{a}(\eta)=\sum_{t=0}^{2m}\alpha_t\eta^t,\\
&\mathrm{a}(\eta)=\sum_{t=1}^{2m}\alpha_t\sum_{l=1}^{t}\eta^{t-l}\omega_{l-1},\\
&\mathscr{R}^{(\sigma)}(\eta)=\sum_{s=1}^{2m}\rho_s^{(\sigma)}\eta^{(s-1)},\\
&r^{(\sigma)}(\eta)=\sum_{s=2}^{2m}\rho_s^{(\sigma)}\sum_{l=1}^{s-1}\eta^{s-l-1}\omega_{l-1},
\end{aligned}$$

where we have set $|\xi|^l\theta_{l-1}=\omega_{l-1}$.

Thus we have

$$\begin{aligned}
q^{(\sigma)}(\eta)&=\Big(\sum_{s=1}^{2m}\rho_s^{(\sigma)}\eta^{s-1}\Big)\Big(\sum_{t=1}^{2m}\alpha_t\sum_{l=1}^{t}\eta^{t-l}\omega_{l-1}\Big)-\Big(\sum_{t=0}^{2m}\alpha_t\eta^t\Big)\Big(\sum_{s=2}^{2m}\rho_s^{(\sigma)}\sum_{l=1}^{s-1}\eta^{s-l-1}\omega_{l-1}\Big)\\
&=\sum_{s=2}^{2m}\sum_{t=1}^{2m}\rho_s^{(\sigma)}\alpha_t\Big(\eta^{s-1}\sum_{l=1}^{t}\eta^{t-l}\omega_{l-1}-\eta^t\sum_{l=1}^{s-1}\eta^{s-l-1}\omega_{l-1}\Big)
\end{aligned}$$

$$+\rho_1^{(\sigma)}\Big(\sum_{t=1}^{2m}\alpha_t\sum_{l=1}^{t}\eta^{t-l}\omega_{l-1}\Big)-\alpha_0\Big(\sum_{s=2}^{2m}\rho_s^{(\sigma)}\sum_{l=1}^{s-1}\eta^{s-l-1}\omega_{l-1}\Big).$$

Since the orders (with respect to η) of the last two terms in the right side are $\leqq 2m-1$, we consider the first term.

If $t>s-1$,

$$\eta^{s-1}\sum_{l=1}^{t}\eta^{t-l}w_{l-1}-\eta^{t}\sum_{l=1}^{s-1}\eta^{s-l-1}\omega_{l-1}=\sum_{l=s}^{t}\eta^{t+s-l-1}\omega_{l-1}.$$

Hence each summand in the first term corresponding to $t>s-1$ is of order $t-1$, $(\leqq 2m-1)$.

Similary, the summands corresponding to $s-1>t$ are of orders $\leqq 2m-2$. Thus the first part of the Lemma is proved.

It is also clear from the above computation that the term involving η^{2m-1} are obtained only when $t=2m$, $l=s$. Hence if we collect all terms of order $2m-1$, we have

$$\alpha_{2m}(\rho_1^{(\sigma)}\omega_0+\rho_2^{(\sigma)}\omega_1+\cdots\cdots+\rho_{2m}^{(\sigma)}\omega_{2m-1})\eta^{2m-1}.$$

Now, $\alpha(\eta)$, $q(\eta)$ have all the properties required in the preceding lemma. Therefore, the conclusion follows at once if we compare the coefficients of the leading terms.

Proof of Lemma 2.5. As before, we put

$$a_k^+(\eta)=a_k^+(\xi,\ \eta),$$

$$b_{j,k}(\eta)=b_{j,k}(\xi,\eta)=\sum_{s=1}^{2m}\beta_s^{(j,k)}\eta^{s-1},$$

$$\mathscr{R}_k^{(\sigma)}(\eta)=\mathscr{R}_{(\sigma)}^{k}(\xi,\ \eta)=\sum_{s=1}^{2m}\rho_k^{(\sigma,k)}\eta^{s-1},$$

$k=1,\cdots\cdots,N;\ j=1,\cdots\cdots,Nm;\ \sigma=1,\cdots\cdots,m.$

Noting that $r_k^{(1)}(\eta)=|\xi|^{m-1}a_k^+(\eta)$ and that ξ is fixed throughtout the discussion, we see the equivalence of the following three statements.

(1) $$\sum_{j=1}^{Nm}\lambda_j b_{j,k}(\eta)\equiv 0\ (\mathit{mod}\ a_k^+(\eta)),\quad k=1,\cdots\cdots,N,$$

(with complex constant λ_j) imply $\lambda_j=0$, $j=1,\cdots\cdots,Nm$,

(2) $$\sum_{j=1}^{Nm}\lambda_j b_{j,k}(\eta)+\sum_{j=(N+k-1)m+1}^{(N+k)m}\lambda_{j,k}\ \mathscr{R}_k^{\ (j-(N+k-1)m,k)}(\eta)=0,\quad k=1,\cdots\cdots,N,$$

(with complex constant λ_j) imply $\lambda_j=0$, $j=1,\cdots\cdots,2Nm$,

(3) $$\sum_{j=1}^{Nm}\lambda_j\beta_s^{(j,k)}+\sum_{j=(N+k-1)m+1}^{(N+k)m}\lambda_{j,k}\rho_s^{(j-(N+k-1)m,k)}=0,$$

$k=1,\cdots\cdots,N;\ s=1,\cdots\cdots,2m,$

(with complex constant λ_j) imply $\lambda_j=0,\ j=1,\cdots\cdots,2Nm.$

In order to see that the first statement implies the second, let us assume (2). Then (1) follows immediately. Consequently by the first statement, we have $\lambda_j=0,\ j=1,\cdots\cdots,Nm,$ and

$$\sum_{j=(N+k-1)m+1}^{(N+k)m}\lambda_j\mathscr{R}_k^{(j-(N+k-1)m)}(\eta)=0,\quad k=1,\cdots\cdots.N.$$

Since, for each $k(k=1,\cdots\cdots,N)$, the polynomials $\mathscr{R}_k^{(\sigma)}(\eta),\ \sigma=1,\cdots\cdots,m,$ are linearly independent, we have $\lambda_j=0,\ j=Nm+1,\cdots\cdots,2Nm.$

Conversely assume (1) holds. Then, noting that the orders of $b_{j,k}(\eta)\leqq 2m-1$, we obtain (2) with suitable constants $\lambda_j,\ j=Nm+1,\cdots\cdots,2Nm.$ This shows that the second statement implies the first. The equivalence of the second and the third statements is obvious. Furthermore, if we compare the coefficient matrices of (3) and (4) below, we see the equivalence of the third statement and the next one which is the conclusion of Lemma 2.5.

$$(4)\qquad \begin{aligned}&\sum_{k=1}^{N}\sum_{s=1}^{2m}\beta_s^{(j,k)}\omega_{s-1}^{(k)}=0,\quad j=1,\cdots\cdots,Nm,\\ &\sum_{s=1}^{2m}\rho_s^{(\sigma,k)}\omega_{s-1}^{(k)}=0,\quad \sigma=1,\cdots\cdots,m;\ k=1,\cdots\cdots,N,\end{aligned}$$

(with complex constant λ_j) imply $\Omega^{(k)}=0,\ k=1,\cdots\cdots,N.$

Indeed the coefficient matrix of (3) is

$$\begin{pmatrix}
\beta_1^{(1,1)}\cdots\cdots\beta_1^{(Nm,1)} & \rho_1^{(1,1)}\cdots\cdots\rho_1^{(m,1)} & & \\
\vdots\qquad\quad\vdots & \vdots\qquad\quad\vdots & & 0\\
\beta_{2m}^{(1,1)}\ \cdots\beta_{2m}^{(Nm,1)} & \rho_{2m}^{(1,1)}\ \cdots\rho_{2m}^{(m,1)} & & \\
\beta_1^{(1,2)}\cdots\cdots\beta_1^{(Nm,2)} & & \ddots & \\
\vdots\qquad\quad\vdots & & & \\
\beta_{2m}^{(1,2)}\ \cdots\beta_{2m}^{(Nm,2)} & & & \\
\vdots\qquad\quad\vdots & & & \\
\beta_1^{(1,N)}\ \cdots\beta_1^{(Nm,N)} & 0 & & \rho_1^{(1,N)}\cdots\cdots\rho_1^{(m,N)}\\
\vdots\qquad\quad\vdots & & & \vdots\qquad\quad\vdots\\
\beta_{2m}^{(1,N)}\ \cdots\beta_{2m}^{(Nm,N)} & & & \rho_{2m}^{(1,N)}\ \cdots\rho_{2m}^{(m,N)}
\end{pmatrix}$$

and that of (4) is its transpose.

PROOF of LEMMA 2.6. Taking a scalar z, let us consider

$$\mathscr{G}(z,\xi)=\int_{-\infty}^{\infty}h(z\xi,\ \eta)d\eta$$

If we set $\eta=z\zeta$, we have

$$\begin{aligned}\mathscr{G}(z\xi)&=z\int_{-\infty}^{\infty}h(z\xi,z\zeta)d\zeta=z^{t+1}\int_{-\infty}^{\infty}h(\xi,\zeta)d\zeta\\ &=z^{t+1}\mathscr{G}(\xi)\end{aligned}$$

Thus the Lemma is proved.

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