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Added Mass for both Vertical and Horizontal Vibration of Two-Dimensional Cylinders of Curvilinear-Element Sections with Chines in a Free Surface

by

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Chine型 船體斷面 柱狀體의 自由水面에서의 上下 및 水平振動에 對한 附加質量

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Chine型船의 船體振動 또는 運動의 解析에 必要한 附加質量計算에 寄與할 目的으로 等角寫像法의 應用, 即 實船과 類似한 斷面形狀을 주는 2 徑數群 寫像函數를 얻어 이를 利用하여 同斷面의 柱狀體에 對하여 理想流體의 自由水面에서의 上下 및 水平振動에 對한 附加質量을 系統的으로 計算했으며, 同計算結果를 上下振動에 對하여서는 Lewis form 柱狀體 및 直線要素斷面柱狀體에 對한 理論的 計算結果와 그리고 水平振動에 對해서는 Lewis form 柱狀體에 對한 것과 比較考察하였다.

Added masses of two-dimensional cylinders of curvilinear-element sections with chines, which are similar to marked *V* character ship sections with either single or double chines, oscillating at high frequency in a free surface of an ideal fluid are calculated for both vertical and horizontal vibration by employing two particular two-parameter families of the conformal transformation. The numerical results are graphically presented in the forms of added mass coefficient curves in terms of the sectional area coefficient and the half beam-draft ratio together with the section contours derived with the employed transformations, and discussed in comparision with those of the Lewis forms and of straightline-element sections with single chine for vertical vibration, and, for horizontal vibration, with those of the Lewis forms.

1. Introduction

The purpose of this work is to give an analytical treatment on calculation of the added mass for both vertical

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** Numbers in Brackets designate References at the End of the Paper.

and horizontal vibration of ships having marked V type section forms with chines. As the mathematical section forms two particular two-parameter families of the conformal transformation are chosen. They are, of course, different from the Lewis forms, and one represents ship sections of single chine and the other those of double chines.

Various methods based on conformal mapping, since F. M. Lewis' work [1]**, have been applied to obtain added masses of ship sections oscillating in a free surface with the relation that a point $P(z)$ on the contour of a double ship section in the complex z-plane can be mapped to a point $P'(\zeta)$ on a unit circle about the origin in ζ-plane (Fig. 1) by the Bieberbach's transformation;

$$z(\zeta)=R[\zeta+\sum_{n=1}^{\infty}a_{2n-1}\zeta^{-(2n-1)}] \tag{1}$$

where

$$z=x+iy \tag{2}$$

$$\zeta=\xi+i\eta=re^{i\theta}=e^{i\theta} \qquad \text{for } r=1.0 \tag{3}$$

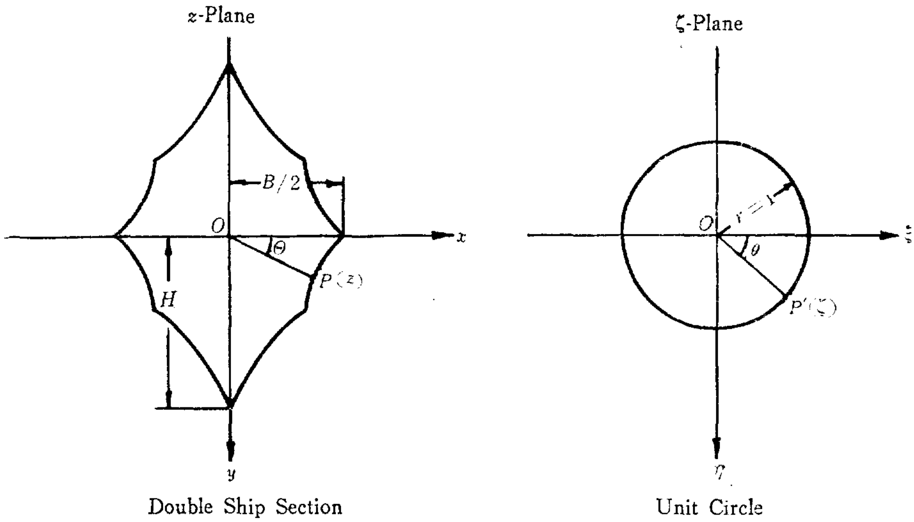


Fig. 1. Conformal Mapping of the Double Ship Section into a Unit Circle.

Notation

A = added mass

a_{2n-1}, a_m = coefficients of the mapping function

B = breadth of ship section

b_n = coefficients of complex potential

$\bar{b}_n$ = complex conjugate of b_n

C = added mass coefficient

$c_n = -ib_n$

g = the gravitational constant

H = draught of ship section

$p=\frac{B}{2H}$; half beam-draft ratio

R = scale factor of the mapping function

r,θ = polar co-ordinates in ζ-plane

S = sectional area of the cylinder

s = arc length along the section contour

T = kinetic energy of fluid surrounding the profile

U, V = horizontal and vertical component of velocity of the profile

$w(z)=\phi+i\psi$; complex potential

$z = x+iy$

$\zeta = \xi+i\eta$

ρ = mass density of fluid

σ = sectional area coefficient

ω = angular frequency of oscillation

Subscripts:

H = horizontal component

V = vertical component

$'$ = for actual ship

and R positive scale factor. The coefficients a_{2n-1}'s are real and with only odd indices because of the symmetry of the section contours about both the vertical and horizontal axes.

Deriving a series of ship-like section contours of a two-parameter family with $a_{2n-1}=a_1$ and a_3, called the Lewis forms, F.M. Lewis [1] calculated two-dimensional added masses for vertical vibration at high frequency in a free surface of an ideal fluid. C.W. Prohaska [2] also investigated some section contours of another two-parameter families with $a_{2n-1}=a_1$ and a_5, a_1 and a_7, and a_3 and a_7 for vertical vibration. L. Landweber and M. Macagno [3] gave a unified treatment on the added mass of the Lewis forms for both vertical and horizontal vibration. They also investigated a three-parameter family characterized with $a_{2n-1}=a_1$, a_3 and a_5 [4] and the technique of conformal mapping to obtain the coefficients of the transformation (1) directly with the aid of high-speed computers [5]. Recently, M. Macagno [6] made a brief comparision of various techniques of calculating the added mass associated with ship vibration.

The works mentioned in the above were mostly concerned with usual ship sections. The Prohaska's work was partly concerned with vertical vibration of unusual section forms such as bulbous bow sections, marked V character sections and marked U character sections. Besides it, Lewis [1] investigated typically unusual rectangular sections and rhombus sections, and K. Wendel [7] rectangular sections with bilge keels. As for the sections with chines, after an investigation of some typical cases [8], J.H. Hwang [9], [10], by employing Schwarz-Christoffel transformation, systematically calculated the added mass of two-dimensional cylinders of straightline-element sections for vertical vibration.

In this work, the author investigated added mass of the curvilinear-element sections with chines for both vertical and horizontal vibration. It is well recognized that section forms concaved, or concaved and convexed slightly, with chines are welcome for medium and high speed boats including planning hulls, and that section forms of the developable hull surface are apt to have slightly convexed shape in the portion below chine line.

2. Formulation of the problem

2.1 Mathematical representation of the section contours

As the mapping function of the two-parameter family characterizing our problems, we take

$$z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_m\zeta^{-m}) \tag{4}$$

with the condition

$$0\leq a_m\leq \frac{1}{m} \tag{5}$$

from the transformation (1), where m is a positive odd integer.

Then it can be easily seen that the mapping function (4) maps the circular section in z-plane into a unit circle in ζ-plane with $a_1=a_m=0$, the elliptical section with $a_m=0$, the hypotrocoidal section not intersect itself with $a_1=0$ and $0<a_m<\frac{1}{m}$, and the hypocycloidal section with $(m+1)$ cusps with $a_1=0$ and $a_m=\frac{1}{m}$. Hence, considering the section contour below waterline and its image, we know that our problems can be characterized with $m=7$ for the single chine type curvilinear-element ship section, and $m=11$ for the double chine type curvilinear-element ship section. Thus we propose to represent mathematically the above mentioned section contours with two particular two-parameter families defined as follows:

$$z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_7\zeta^{-7}) \tag{6}$$

$$z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_{11}\zeta^{-11}) \tag{7}$$

To obtain useful section contours the condition

$$\left.\begin{array}{l}\mathrm{Re}\{z(0)\}\geq \mathrm{Re}\{z(\theta)\}\geq 0\\ 0\leq \mathrm{Im}\{z(\theta)\}\leq \mathrm{Im}\{z(\pi/2)\}\end{array}\right\};\qquad 0\leq\theta\leq\frac{\pi}{2} \tag{8}$$

must be satisfied. Form Equation (8) it is found that the condition of constraints on a_1 should be

$$|a_1|\leq(1-ma_m);\qquad m=7 \text{ or } 11 \tag{9}$$

Now the condition (5) can be specified as follows:

$$\left.\begin{array}{l}0\leq a_m\leq\frac{1}{m},\ \text{when } a_1=0\\ 0\leq a_m<\frac{1}{m},\ \text{when } a_1\neq 0\end{array}\right\};\qquad m=7 \text{ or } 11 \tag{10}$$

The extreme cases of the available section contours in z-plane with the mapping functions (6) and (7) and the conditions (9) and (10) are clearly characterized with $a_m=0$ (circles) and $a_m=\frac{1}{m}$ (hypocycloids) in case of $a_1=0$, and with $a_m=0$ (elliptical sections) and $a_m=1/(m+\alpha)$ in case of $a_1\neq0$. The values of α depend on the beam-draft ratio and will be discussed further later on.

Refering to Equations (2) and (3), we can write the transformation (6) and (7) in the parametric form of θ;

$$\left.\begin{array}{l}x=R[(1+a_1)\cos\theta+a_m\cos m\theta]\\ y=R[(1-a_1)\sin\theta-a_m\sin m\theta]\end{array}\right\};\qquad m=7 \text{ or } 11 \tag{11}$$

where R is, for convenience, to be taken as

$$R=(1+a_1+a_m)^{-1};\qquad m=7 \text{ or } 11 \tag{12}$$

so that the half beam, $|x|_{max}$ or $B/2$, of the section contours in z-plane may always become unity. And the half beam-draft ratio p is obtained by

$$p=\frac{B}{2H}=\frac{1+a_1+a_m}{1-a_1+a_m};\qquad m=7 \text{ or } 11 \tag{13}$$

where H denotes draught of the section.

From Equation (13), for given values of p, we have

$$a_1=\left(\frac{p-1}{p+1}\right)(1+a_m);\qquad m=7 \text{ or } 11 \tag{14}$$

and the condition (10) becomes

$$\left.\begin{array}{ll}0\leq a_m\leq\frac{1}{m}, & \text{when } p=1,\ \text{i.e. } a_1=0\\ 0\leq a_m<\dfrac{1}{\frac{1}{2}(p-1)(m+1)+m}, & \text{when } p\neq1,\ \text{i.e. } a_1\neq0\end{array}\right\};\qquad m=7 \text{ or } 11 \tag{15}$$

From Equations (11), and (13) or (14) it is easily observed that, for given values of m and a_m, the section contours corresponding to $a_1=-\beta$ are those obtainable merely by rotating those derived with $a_1=+\beta$ through 90 drgree, and that their beam-draft ratios are in inverse relation with each other. The Equation (14) and conditions (9) and (15) are graphically shown in Fig. 2 for $m=7$ and in Fig. 3 for $m=11$.

The sectional area S below waterline of the section contours obtained by Equation (11) will be turned out to be

$$S=\frac{1}{2}\oint x\,dy=\frac{\pi}{2}f_a \tag{16}$$

and the sectional area coefficient σ

$$\sigma=\frac{\pi}{4}\frac{(1-a_1^2-ma_m^2)}{(1+a_1+a_m)(1-a_1+a_m)}=\frac{\pi}{4}p\,f_a \tag{17}$$

where

$$f_a=R^2(1-a_1^2-ma_m^2);\qquad m=7 \text{ or } 11 \tag{18}$$

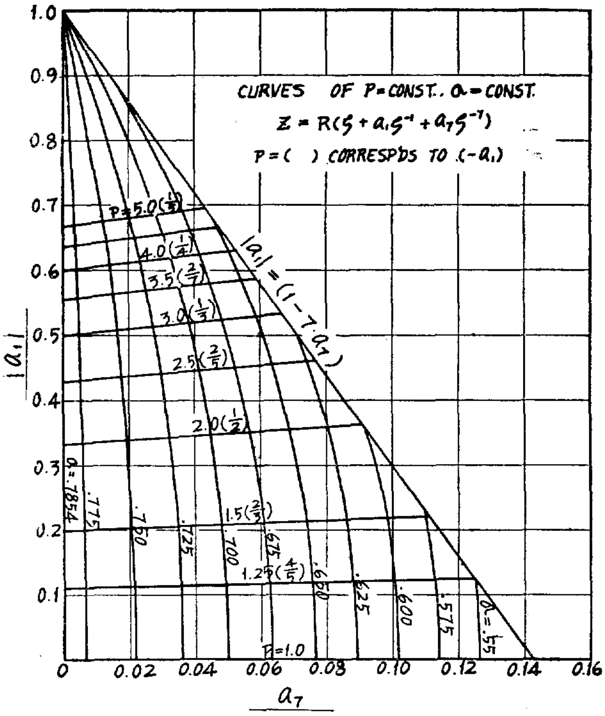


Fig. 2. Curves of p=const. and σ=const. in Terms of a_1 and a_7

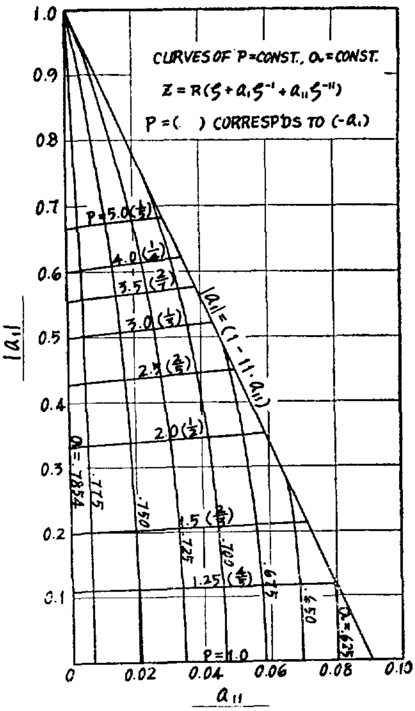


Fig. 3. Curves of p=const. and σ=const. in Terms of a_1 and a_{11}

Since Equation (16) is after the scale factor, it should be interpreted for the actual area of ship sections S', half-beam of which is not unity but $B/2$, as follows:

$$S'=\left(\frac{B}{2}\right)^2 S=\left(\frac{B}{2}\right)^2 \frac{2\sigma}{p} \tag{19}$$

The curves of σ=const. based on Equation (17) are also shown in Figs. 2 and 3 in terms of a_1 and a_m together the curves of p=const..

From Equations (9), (14), (15) and (17), it is clear that the maximum value of σ will always become

$$\sigma_{\max}=\frac{\pi}{4} \tag{20}$$

with $a_m=0$ which corresponds to an elliptical section, and that, for given values of p, the minimum value of σ imits to

$$\sigma_{\min}=\frac{\pi}{4} p \frac{\left[1-\left(\frac{p-1}{p+1}\right)^2(1+(a_m)_{\max})^2-m(a_m)^2{}_{\max}\right]}{\left[1+\left(\frac{p-1}{p+1}\right)(1+(a_m)_{\max})+(a_m)_{\max}\right]^2} \tag{21}$$

when a_m limits to

$$(a_m)_{\max}=\frac{1}{\frac{1}{2}(p-1)(m+1)+m} \tag{22}$$

where $m=7$ or 11. For each value of p the limitation of the range of σ is shown in Table 1 together with the corresponding value of a_m.

With known values of p and σ which lie within the above range, we can find out the corresponding values of a_1 and a_m either from Fig. 2 and Fig. 3, or from Equations (13) and (17).

Table 1. Limits of a_m and σ

p		$z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_7\zeta^{-7})$				$z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_{11}\zeta^{-11})$			
		outer limit		inner limit		outer limit		inner limit	
		a_7	σ_{max}	a_7	σ_{min}	a_{11}	σ_{max}	a_{11}	σ_{min}
1.00				1/7	0.5154			1/11	0.6
1.25	0.80			0.12495	0.5499			0.08	0.6240
1.50	2/3			0.11111	0.5727			0.07142	0.6400
2.00	0.50			0.09090	0.6014			0.05882	0.6600
2.50	0.40	0	0.7854	0.07692	0.6185	0	0.7854	0.05	0.6720
3.00	1/3	(all an elliptical		0.06666	0.6300	(all an elliptical		0.04347	0.6800
3.50	2/7	section)		0.05882	0.6382	section)		0.03846	0.6857
4.00	0.25			0.05263	0.6443			0.03448	0.6899
5.00	0.20			0.04347	0.6529			0.02857	0.6959

2.2 Complex potential and boundary conditions

We assume that the complex potential $w(z)$ which will satisfy the boundary conditions presented may be obtained in parametric form together with Equation (1) for both vertical and horizontal vibration;

$$w(z)=\phi+i\psi=\sum_{n=1}^{\infty} b_n\zeta^{-n} \tag{23}$$

or

$$\left.\begin{aligned}\phi&=\frac{1}{2}\sum_{n=1}^{\infty} r^{-n}(b_n e^{-in\theta}+\bar{b}_n e^{in\theta})\\ \psi&=\frac{1}{2i}\sum_{n=1}^{\infty} r^{-n}(b_n e^{-in\theta}-\bar{b}_n e^{in\theta})\end{aligned}\right\} \tag{24}$$

where ϕ and ψ are velocity potential and stream function refered to the flow around the oscillating two-dimensional cylinder, respectively.

In case that a body oscillates at one of its principal modes or in simple harmonic motion with a small amplitude in a free surface, the boundary condition on the free surface becomes [11]

$$\omega^2\phi=g\frac{\partial\phi}{\partial y} \tag{25}$$

where ω is an angular frequency and g the gravitational constant. Hence, in case of high frequency, $\omega\to\infty$, the boundary condition (25) on the free surface turns out to be

$$\phi=0 \tag{26}$$

The boundary condition at infinity is also satisfied by the Equation (26).

In case of vertical vibration, the body boundary condition can be satisfied by supposing that the double section oscillates as a single rigid form. Then the boundary condition on the double section turns out to be (Appendix 1)

$$\psi=-Vx \tag{27}$$

where V is the velocity in y-direction.

In case of horizontal vibration, the body boundary condion can be satisfied by supposing that the upper and lower halves of the double section instantaneously have velocities of same magnitude but opposed direction. Then the boundary condition on the double section turns out to be (Appendix 1)

$$\psi=U|y| \tag{28}$$

where U is the velocity in x-direction.

2.3 Kinetic energy of the fluid and added mass

The kinetic energy T of a fluid surrounding an oscillating body is calculated by

$$T=-\frac{\rho}{2}\oint\phi\, d\psi \tag{29}$$

where ρ is the mass density of the fluid and the integral extends over all the boundaries of the fluid.

Refering to Equations (23) or (24), we write Equation (29) in parametric form of θ, and may evaluate the integral over the unit circle in ζ-plane;

$$T=-\frac{\rho}{2}\int_0^{2\pi}\phi\frac{\partial\psi}{\partial\theta}d\theta \tag{30}$$

However, the integral vanishes over the free surface in both cases of vertical and horizontal vibration because of the condition $\phi=0$ on that boundary. Therefore the kinetic energy T' of the fluid below the free surface, that is, of the actual ship section, is half of that given by Equation (30) and turns out to be (Appendix 2)

$$T'=\frac{\pi\rho}{4}\sum_{n=1}^{\infty}n\,|b_n|^2 \tag{31}$$

Since the added mass A per unit length of the cylinder can be defined by

$$A=\frac{T'}{\frac{1}{2}\,(\text{velocity})^2}, \tag{32}$$

the added mass coefficient C refered to the Lewis' definition and to the scale factor R which makes the length of a semi-axis to be taken as the basis, $B/2$ or H, be unity becomes

$$C=\frac{A}{(\pi\rho/2)}=\frac{4T'}{\pi\rho\,(\text{veloctity})^2} \tag{33}$$

Hence, we obtain for vertical vibration

$$A_V=\frac{\pi\rho}{2V^2}\sum_{n=1}^{\infty}n\,|b_n|^2 \tag{34}$$

$$C_V=\frac{1}{V^2}\sum_{n=1}^{\infty}n\,|b_n|^2 \tag{35}$$

and for horizontal vibration

$$A_H=\frac{\pi\rho}{2U^2}\sum_{n=1}^{\infty}n\,|b_n|^2 \tag{36}$$

$$C_H=\frac{1}{U^2}\sum_{n=1}^{\infty}n\,|b_n|^2 \tag{37}$$

2.4 Added mass for vertical vibration

Substituting the right hand sides of Equations (11) and (24) for x and ψ of the boundary condition (27), we obtain for the unit circle

$$\left.\begin{array}{ll} b_1=-iRV(1+a_1) & \\ b_m=-iRVa_m\,; & m=7 \text{ or } 11 \\ \text{All other } b_n\text{'s are zero} & \end{array}\right\} \tag{38}$$

Hence, from Equations (34) and (35), the added mass and added mass coefficient for vertical vibration turn out to be

$$A_V=\frac{\pi\rho}{2}R^2(1+2a_1+a_1^2+7a_7^2) \tag{39}$$

$$C_V=R^2(1+2a_1+a_1^2+7a_7^2) \tag{40}$$

for $m=7$ which represents the single chine type section contours, and

$$A_V=\frac{\pi\rho}{2}R^2(1+2a_1+a_1^2+11a_{11}^2) \tag{41}$$

$$C_V=R^2(1+2a_1+a_1^2+11a_{11}^2) \tag{42}$$

for $m=11$ which represents the doubler chine type section contours.

For the calculation of C_V, the scale factor R is to be taken as Equation (12), so that the added mass A'_V of the section contour before the scale factor, that is of the actual ship section half-breadth at free surface of which is not unity but $B/2$, should be calculated by

$$A'_V=C_V\frac{\pi\rho}{2}\left(\frac{B}{2}\right)^2 \tag{43}$$

2.5 Added mass for horizontal vibration

From Equation (24) and the boundary condition (28), we know that ψ should be an even function of θ and b_n imaginary. Hence, we have ψ represented by the Fourier cosine series;

$$\psi=U|y|=\sum_{n=1}^{\infty} c_n \cos n\theta \tag{44}$$

where $\quad b_n=ic_n.$ (45)

Then, in case of our problem, coefficient c_n can be obtained by

$$c_n=\frac{2RU}{\pi}\int_0^{\pi}\{(1-a_1)\sin\theta-a_m\sin m\theta\}\cos n\theta\, d\theta\,; \qquad m=7 \text{ or } 11 \tag{46}$$

by virture of Equation (11).

Evaluation of the above integral shows that

$$\left.\begin{aligned} &c_{2n+1}=0\,; \qquad n=0,1,2,3,\cdots\cdots \\ &c_{2n}=-\frac{4RU}{\pi}\left(\frac{a_1-1}{1-4n^2}+\frac{ma_m}{m^2-4n^2}\right);\ n=1,2,3,\cdots\cdots \end{aligned}\right\}; \qquad m=7 \text{ or } 11 \tag{47}$$

Hence, by virture of Equations (36), (37) and (45), the added mass and the added mass coefficient for horizontal vibration can be calculated by

$$A_H=\frac{8\rho R^2}{\pi}\sum_{n=1}^{\infty}(2n)\left(\frac{a_1-1}{1-4n^2}+\frac{ma_m}{m^2-4n^2}\right)^2; \qquad m=7 \text{ or } 11 \tag{48}$$

$$C_H=\frac{16R^2}{\pi^2}\sum_{n=1}^{\infty}(2n)\left(\frac{a_1-1}{1-4n^2}+\frac{ma_m}{m^2-4n^2}\right)^2; \qquad m=7 \text{ or } 11 \tag{49}$$

Evaluating the Equation (48) and (49), we obtain

$$A_H=\frac{8\rho R^2}{\pi}\{\alpha_{11}(1-a_1)^2+2\alpha_{17}(1-a_1)a_7+\alpha_{77}a_7^2\} \tag{50}$$

$$C_H=\frac{16R^2}{\pi^2}\{\alpha_{11}(1-a_1)^2+2\alpha_{17}(1-a_1)a_7+\alpha_{77}a_7^2\} \tag{51}$$

where $\alpha_{11}=0.25$, $\alpha_{17}=0.161111\cdots$, and $\alpha_{77}=4.064602\cdots$ (Appendix 3), for $m=7$ which represents the single chine type section, and

$$A_H=\frac{8\rho R^2}{\pi}\{\alpha_{11}(1-a_1)^2+2\alpha_{1\cdot 11}(1-a_1)a_{11}+\alpha_{11\cdot 11}a_{11}^2\} \tag{52}$$

$$C_H=\frac{16R^2}{\pi^2}\{\alpha_{11}(1-a_1)^2+2\alpha_{1\cdot 11}(1-a_1)a_{11}+\alpha_{11\cdot 11}a_{11}^2\} \tag{53}$$

where $\alpha_{11}=0.25$, $\alpha_{1\cdot 11}=0.122169\cdots$, and $\alpha_{11\cdot 11}=6.533984\cdots$ (Appendix 3), for $m=11$ which represents the double chine type section.

The scale factor R for this time is to be taken as

$$R=(1-a_1+a_m)^{-1}; \qquad m=7 \text{ or } 11 \tag{54}$$

so that the added mass A'_H of the section contours before the scale factor, that is of the actual ship section draught of which is not unity but H, should be calculated by

$$A'_H = C_H \frac{\pi \rho}{2} H^2 \tag{55}$$

The added mass coefficient C for both vertical and horizontal vibration may also be calculated directly by employing p and σ as shown in Appendix 4.

3. Numerical results and discussion

The section contours obtained by the transformation (6) and (7), that is Equation (11), together with the conditions of constraints (9) and (10) are shown in Fig. 4 for $m=7$ and in Fig. 5 for $m=11$. Those contours are originally of positive values of a_1 obtained from Equation (14) by substituting the half beam-draft ratio of

$p=1.00,\ 1.25,\ 1.50,\ 2.00,\ 2.50,\ 3.00,\ 3.50,\ 4.00,$ and 5.00

for p and the value of a_m from 0 to the permissible maximum value (15) with an interval of 0.01 for a_m.

Rotating the above contours through 90 degree in clockwise sense after x- and y-axis exchanged, we can now have the section contours of the inverse values of the above half beam-draft ratio, that is,

$p=1.00,\ 0.80,\ 2/3,\ 0.50,\ 0.40,\ 1/3,\ 2/7,\ 0.25$ and 0.20

in the same sequence as above. These contours, as mentioned already in the section 2.1, are those obtainable from the same mapping function only by substituting negative values of a_1, absolute values of which are same as those used in derivation of the former original contours. This relation can be easily verified by substituting $1/p$ for p in Equation (14) and inspecting Equation (11).

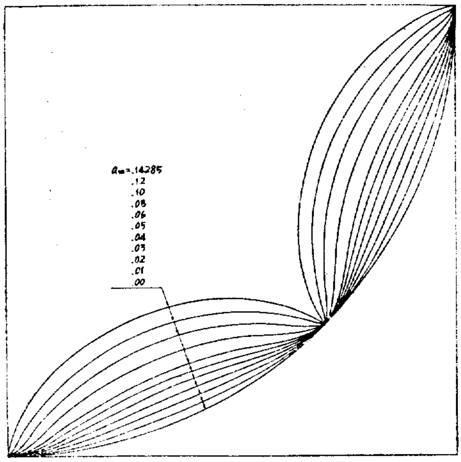
For all the section contours mapped in Fig. 4 and Fig. 5, σ, C_V and C_H are calculated. The results are tabulated in Table 3 for $m=7$ and in Table 4 for $m=11$ together with the corresponding values of a_m and a_1. From those tables, curves of $C_V=$const. and $C_H=$const. are compiled in terms of p and σ, and are shown in Fig. 6 for $m=7$ and in Fig. 7 for $m=11$.

Comparing the numerical results on C_V with those of the Lewis forms and the Prohaska's compiled data [2] or the Landweber and Macagno's [3], we can clearly recognize that the curvilinear-element sections with chines give considerably greater values of C_V than the Lewis forms for the same values of p and σ. It is observed that the smaller σ has, for a given value of p, the bigger rate of increment of C_V over that of the Lewis forms, and that the smaller p has, for a given value of σ, the bigger rate of increment of C_V over that of the Lewis forms. It is also found that the rate of increment of C_V of the double-chine type sections over the Lewis forms is almost twice that of the single-chine type sections for the given values of p and σ.

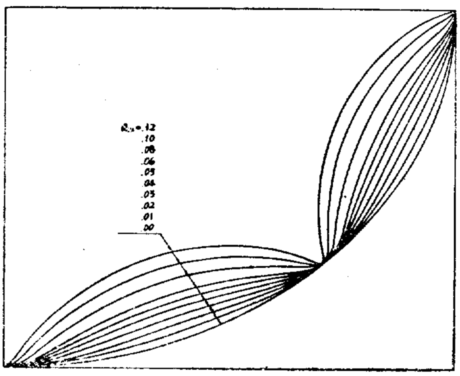
Numerically speaking in general, the rate increment of C_V of the curvilinear-element sections over the Lewis forms is distributed, in case of p below 1.0, between about 5 per cent and well over 10 per cent for σ below 0.7 and within about 5 per cent for σ over 0.7. And it is,in case of p over 1.0, distributed between a few per cent and a little more than 5 per cent for σ below 0.7 and within a few per cent for σ over 0.7.

To show the above mentioned clearly, C_V curves of the Lewis forms and the curvilinear-element sections with chines are graphically compared in Fig. 8 for the cases of $p=5/3$, 1.00 and 0.20. In that figure, C_V curves of the straightline-element sectins with single chine from Hwang's work [9] are also shown for the cases of $p=5/3$ and 1.00. Hwang's numerical calculation did not cover $p=0.20$.

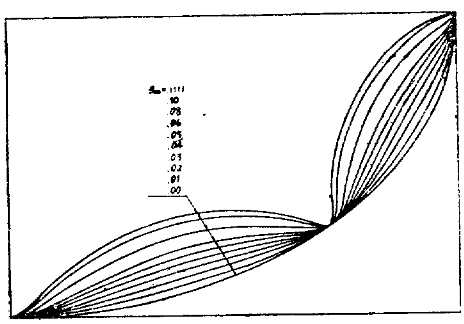
After an investigation of the added mass for vertical vibration of some typical sections of marked V character, similar to the author's series of $m=7$, Prohaska [2] found that, for given values of p and σ, V type sections always give a greater C_V-value than U type sections, and suggested that, for sections of marked V or U character, the value of C_V from his compiled data basically for the Lewis forms might be corrected by ± 5 per cent. Hwang's work [9] for the vertical vibration of the straightline-element sections with single chine also showed the same tendency in general, but the difference of magnitude of C_V is well over the Prohaska's



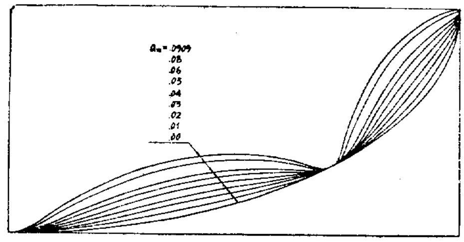
$p=1.00$



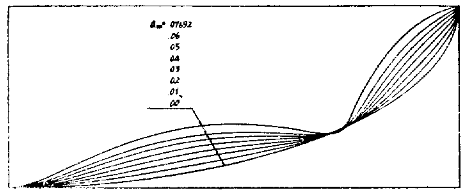
$p=1.25$, and (0.80)



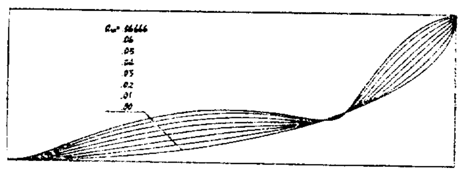
$p=1.50$, and (2/3)



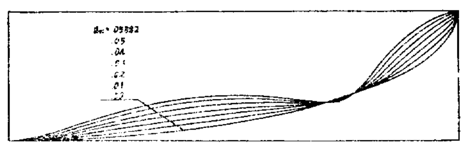
$p=2.00$, and (0.50)



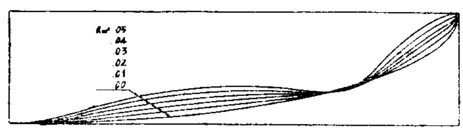
$p=2.50$, and (0.40)



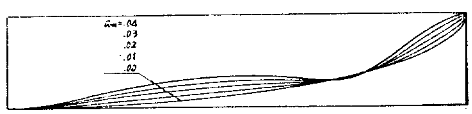
$p=3.00$, and (1/3)



$p=3.50$, and (2/7)



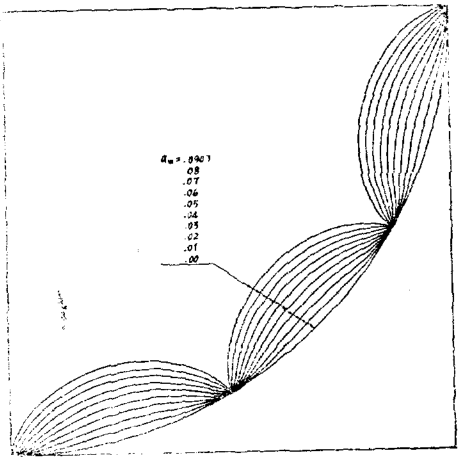
$p=4.00$, and (0.25)



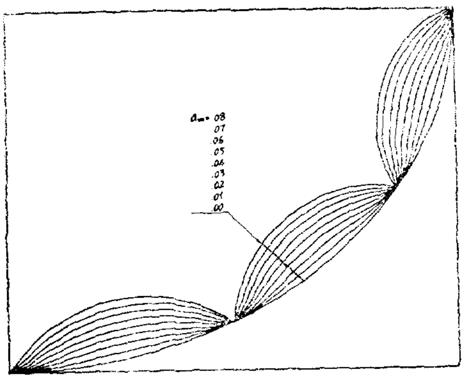
$p=5.00$, and (0.20)

Fig. 4. Section Contours derived with the Transformation $z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_7\zeta^{-7})$

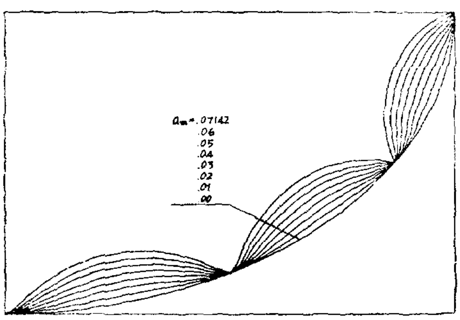
$p=($ $)$ Corresponds to the Contour Rotated Clockwise thru. 90 deg.



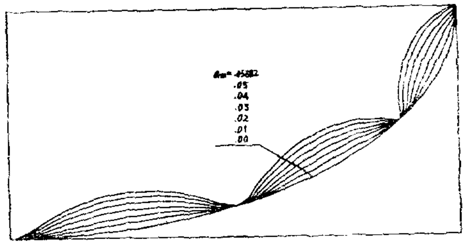
$p=1.00$



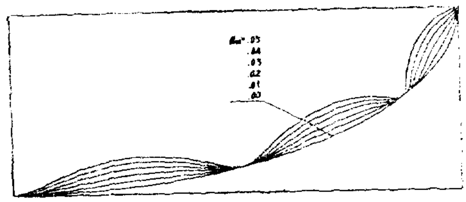
$p=1.25$, and (0.80)



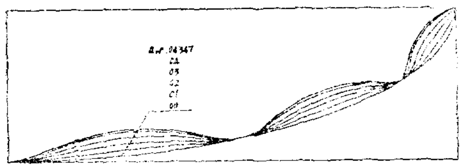
$p=1.50$, and (2/3)



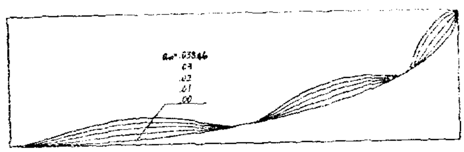
$p=2.00$, and (0.5)



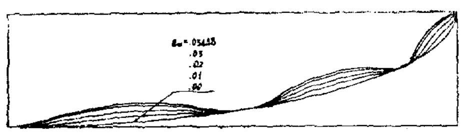
$p=2.50$, and (0.40)



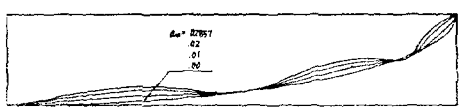
$p=3.00$, and (1/3)



$p=3.50$, and (2/7)



$p=4.00$, and (0.25)



$p=5.00$, and (0.20)

Fig. 5. Section Contours derived with the Transformation $z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_{11}\zeta^{-11})$

$p=(\quad)$ Corresponds to the Contour Rotated Clockwise thru. 90 deg.

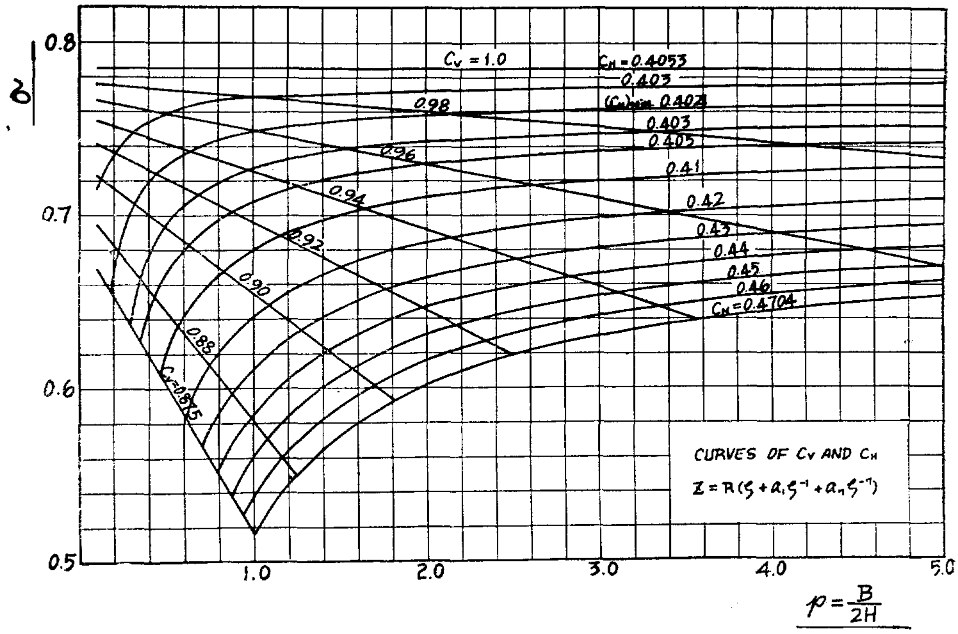


Fig. 6. Curves of C_V and C_H in Terms of p and σ : $z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_7\zeta^{-7})$

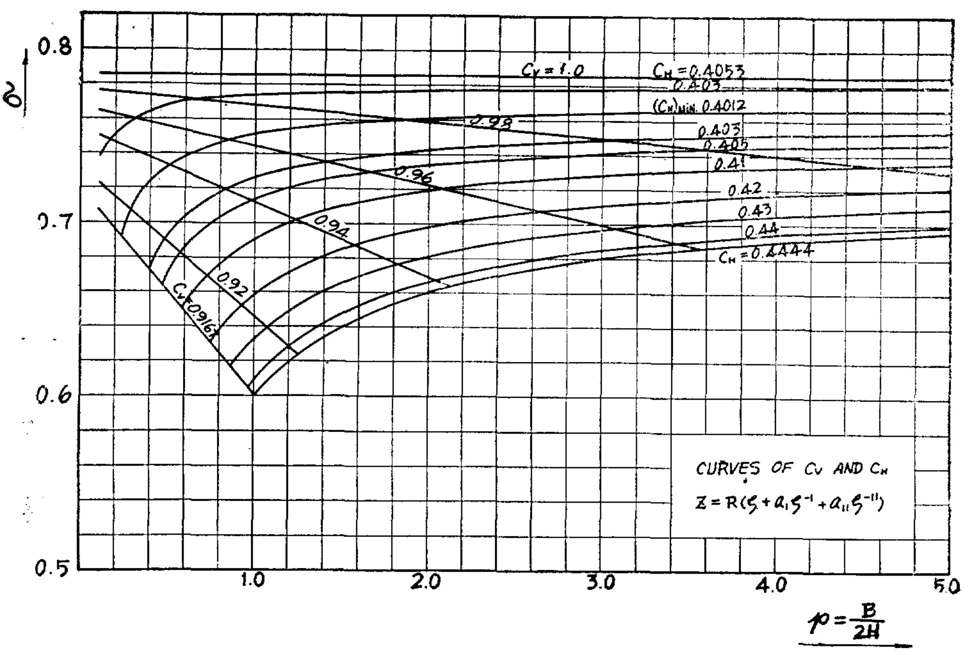


Fig. 7. Curves of C_V and C_H in Terms of p and σ : $z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_{11}\zeta^{-11})$

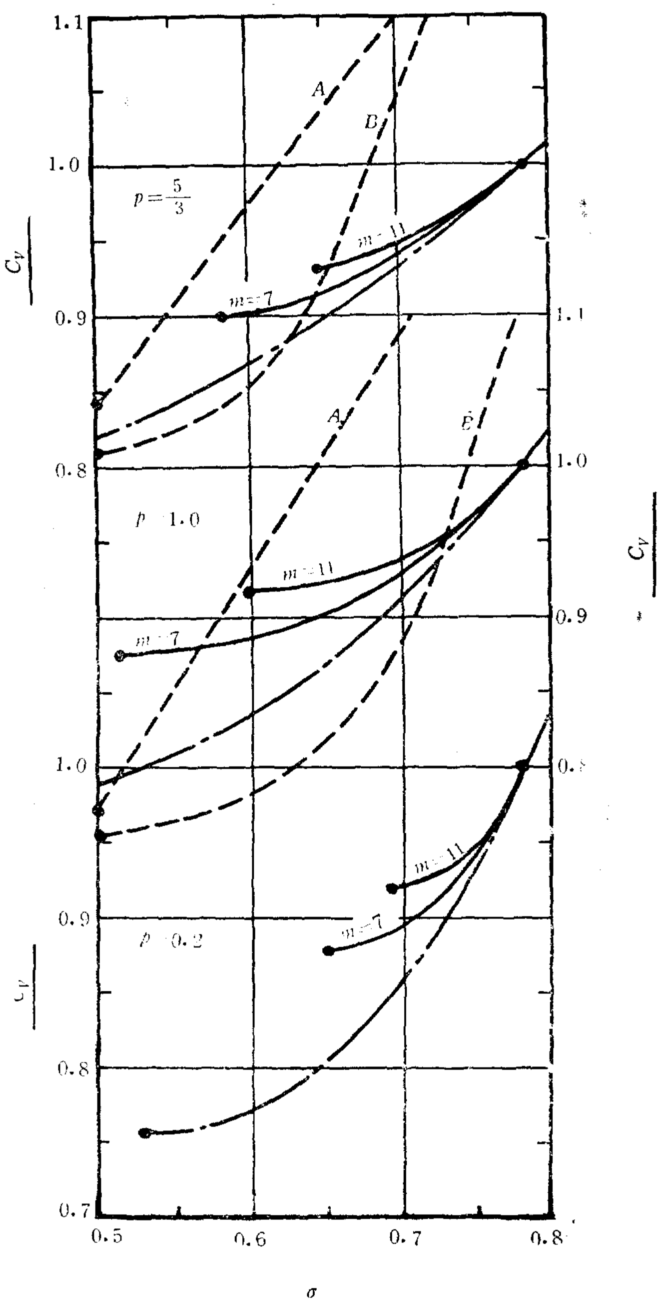


Fig. 8. Comparision of C_V with those of the Lewis Forms and of the Straightline Sections with Single Chine

——— Curvilinear-element sec. w/chines

—·— Lewis forms

– – – – Straightline-element sec. w/single chine; *A*: vertical side, *B*: raked side and bottom-deadrise angle of 0.1 π

• Limits of σ

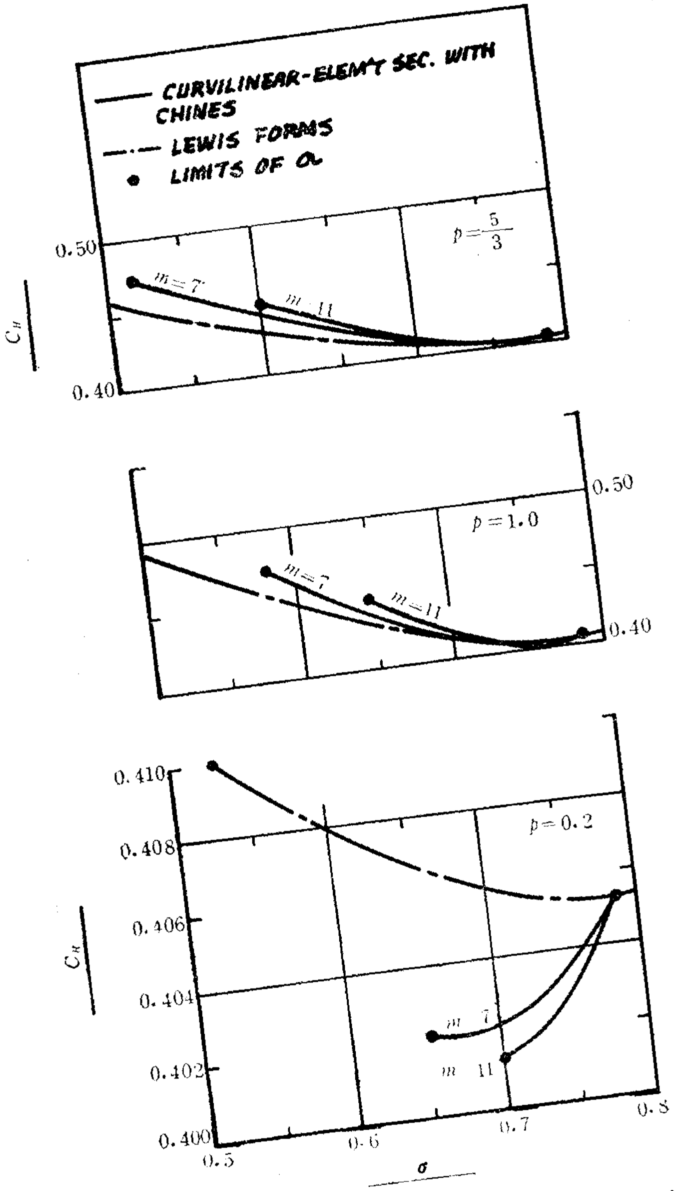


Fig. 9. Comparision of C_H with those of the Lewis Forms

suggestion as shown in Fig. 8. Furthermore, from Hwang's work we can find out two significant points the straightline-element sections give markably larger values of C_V than the Lewis forms as σ and bo deadrise angle increase, and give markably smaller values of C_V as σ and bottom-deadrise angle dec and that, for given value of σ and p, the larger bottom-deadrise angle gives the larger values of C_V. S the straightline-element sections the magnitude of angle at chine and of bottom-deadrise is the sign factor influencing on the magnitude of C_V.

As to horizontal vibration, since no other works similar to Prohaska's or Hwang's works on vibration are available, the comparison is made only with the Lewis forms. Here, a significant that the curvilnear-element sections with chines give the minimum value of C_H at σ less than 0.785 corresponds to an elliptical section, The σ-value for $(C_H)_{min}$ depends on p against the Lewis forms at σ

Table. 2. a_m and σ for $(C_H)_{\min}$

p	$z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_7\zeta^{-7})$		$z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_{11}\zeta^{-11})$	
	a_7	σ	a_{11}	σ
0.20	0.0385	0.6690	0.0337	(0.6783)*
0.25	0.0369	0.6887	0.0323	0.6967
2/7	0.0358	0.6986	0.0314	0.7059
1/3	0.0345	0.7085	0.0302	0.7150
0.40	0.0328	0.7184	0.0287	0.7242
0.50	0.0303	0.7283	0.0268	0.7334
2/3	0.0274	0.7382	0.0240	0.7425
0.80	0.0253	0.7432	0.0222	0.7471
1.00	0.0228	0.7481	0.0200	0.7517
1.25	0.0202	0.7521	0.0177	0.1554
1.50	0.0181	0.7547	0.0159	0.7578
2.00	0.0151	0.7580	0.0132	0.7600
2.50	0.0129	0.7600	0.0113	0.7627
3.00	0.0113	0.7613	0.0039	0.7639
3.50	0.0100	0.7623	0.0038	0.7648
4.00	0.0090	0.7630	0.0079	0.7655
5.00	0.0075	0.7640	0.0066	0.7664

( )* beyond the applicable range

in all cases of p. In Figs. 6 and 7, $(C_H)_{\min}$ curves are of those obtained analytically. For given p, a_m for $(C_H)_{\min}$, $(a_m)_p$, is found to be

$$(a_m)_p=\frac{(\alpha_{11}-\alpha_{1m})\left(1-\frac{p-1}{p+1}\right)}{(\alpha_{11}-\alpha_{1m})\left(\frac{p-1}{p+1}\right)-(\alpha_{1m}-\alpha_{mm})}; \qquad m=7 \text{ or } 11 \tag{53}$$

from

$$\frac{dC_H}{da_m}=0$$

And from Equations (14), (51) and (53) we know that $(C_H)_{\min}$ becomes

$$(C_H)_{\min}\fallingdotseq\begin{cases}0.4021 & \text{for } m=7\\ 0.4012 & \text{for } m=11\end{cases}$$

in all cases of p. In Table 2, the values of a_m and σ giving $(C_H)_{\min}$ are shown for each value of p. It should be noted that the Lewis forms give $(C_H)_{\min}=0.4053$ at $a_3=0$, elliptical sections, in all cases of p due to $\alpha_{11}=\alpha_{13}$ in Equation (56).

As shown in Fig. 9, the difference of C_H between the author's sections and the Lewis forms is not so much significant as that of C_V; below 5 per cent. The tendency that the smaller σ, for given p, has the greater difference is same as that of C_V. However, for given σ, the greater p gives the more difference. Those sections give smaller values of C_H than the Lewis forms in cases of both p below 0.4 and σ around those giving minimum values of C_H, and larger values in other cases.

As for the three-dimensional correction factor for calculation of the virtual inertia coefficient by employing two-dimensionally calculated added mass, the works of F.M. Lewis [1] and J. Lockwood Taylor [12] may be useful for vertical vibration of such fine ships, and with respect to horizontal vibration, the works of T. Kumai [13] and J. Lockwood Taylor [14] are available. However, proper three-dimensional correction factors for chine-type hull forms may have to be investigated.

4. Conclusion

To contribute towards more accurate calculation of the added mass for the flexural vibrations of ships having marked V character sections with chines, two kinds of series of the mathematical section corresponding to single chine type and double chine type each are derived by employing two particular two-parameter families of the conformal transformation, and two-dimensional added mass of those sections are systematically calculated for both vertical and horizontal vibration at high frequency in a free surface of an ideal fluid.

The numerical results on vertical vibration show that the curvilinear-element sections with chines give markably different values of the added mass coefficient from those of the Lewis forms. Prohaska's suggestion on calculation of added mass for vertical vibration of marked V charater sections based on his general observation [2] seems not to be of the generality applicable to V character sections with chines. For more precise calculation of added mass for vertical oscillation of chine type sections, the author's work and the Hwang's work, or either one, depending on the character of the section, will prove to be beneficial.

The numerical results on horizontal vibration show that the curvilinear-element sections give smaller values of the added mass coefficient than the Lewis forms in cases of both p below 0.4 and σ around those giving minimum values of C_H, and larger values in other cases. In any cases, the difference is not so much as that in the case of vertical vibration. Since the case of horizontal vibration of chine type sections, as far as the author awares, has not been handled previously, this work will also prove to be beneficial for the treatment of ship vibration in that mode.

Acknowledgment

The author gratefully acknowledges the guidance of Prof. Jong-Heul Hwang of the Seoul National University throughout this work. He is also indebted to Dr. Hun Chul Kim, Korean Institute of Science and Technology, and to Prof. Nack-Joo Lee, Seoul National University, for their helpful suggestions for the solution of problems of various aspects. And this work was supported by the Ministry of Education. Upon this opportunity he wishes to express his sincere thanks to them. Finally, he expresses his thanks to Mr. Y. J. Kwon and the students who gave wonderful assistance to him for numerical calculation and graphing.

References

[1] F.M. Lewis: "The Inertia of the Water Surrounding a Vibrating Ship", *Trans. SNAME*, Vol. 37, 1929

[2] C.W. Prohaska: "Vibrations Verticales Du Navire", *Bulletin de L'Association Technique Maritime et Aéronautique*, 1947

[3] L. Landweber and M. Macagno: "Added Mass of Two-Dimensional Forms Oscillating in a Free Surface", *Journal of Ship Research*, Vol. 1, No. 3., *SNAME*, 1957

[4] L. Landweber and M. Macagno: "Added Mass of a Three-Parameter Family of Two-Dimensional Forms Oscillating in a Free Surface" *Journal of Ship Research*, Vol. 2, No. 4, *SNAME*, 1959

[5] L. Landweber and M. Macagno: "Added Masses of Two-Dimensional Forms by Conformal Mapping", *Journal of Ship Research*, Vol. 11, No. 2, *SNAME*, 1967

[6] M. Macagno: "A Comparision of Three Methods for Computing the Added Mass of Ship Sections," *Journal of Ship Research*, Vol. 12, No. 4, *SNAME*, 1968

[7] K. Wendel: "Hydrodynamische Massen und Hydrodynamische Massenträgheitsmomente," *Jahrb. d. Schiffbautechn. Ges.*, Vol. 44, 1950

[8] J.H. Hwang and K.C. Kim: "A Note on the Added Mass Associated with the Vertical Oscillation of the

Straight-Framed Ships in a Free Surface," *Science and Engineering Report*, Vol. 3, No. 1, *College of Engineering, SNU*, 1968 (written in Korean).

[9] J. H. Hwang: "Added Mass of Two Dimensional Cylinders with the Sections of Straight Frames Oscillating Vertically in a Free Surface," *Journal of the Society of Naval Architects of Korea*, Vol. 5, No. 2, 1968

[10] J.H. Hwang: "The Analysis on the Added Mass of Two Dimensional Cylinders of Four and Five Parameter Families of Straight-framed Sections Oscillating Vertically in a Free Surface," *Journal of the Korean Society of Mechanical Engineers*, Vol. 8, No. 4, 1968

[11] H. Lamb: *Hydrodynamics* (Art. 227), 6th Edition, Dover Publications, 1945

[12] J. Lockwood Taylor: "Some Hydrodynamical Inertia Coefficients," *Phil. Mag.*, Vol. 9, 1930

[13] T. Kumai: "Some Correction Factors for Virtual Inertia Coefficient on the Horizontal Vibration of a Ship," *Journal of the Society of Naval Architects of Japan*, Vol. 108, 1960

[14] J. Lockwood Taylor: "Vibration of Ships," *Trans. Royal Institute of Naval Architects*, Vol. 72, 1930

Appendix 1

In case of vertical vibration of the double section, the body boundary condition, which also at the time satisfies the condition (26) on the free surface is

$$\frac{\partial \phi}{\partial n} = V\frac{\partial y}{\partial n}$$

where n is the direction of the outward normal to the double section. In addition we have the relations

$$\frac{\partial y}{\partial n} = -\frac{\partial x}{\partial s}$$

and, by the Cauchy-Riemann equations,

$$\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial s}$$

where s denotes arc length positive in the clockwise sense along the double section. Thus we can write

$$\frac{\partial \psi}{\partial s} = -V\frac{\partial x}{\partial s}$$

and obtain the boundary condition (27) on the double section.

In case of horizontal vibration of the double section, the boundary condition on the body may be assumed to be

$$\frac{\partial \phi}{\partial n} = U\frac{\partial x}{\partial n}; \qquad y>0$$

$$\frac{\partial \phi}{\partial n} = -U\frac{\partial x}{\partial n}; \quad y<0$$

Furthermore, we have the relations

$$\frac{\partial x}{\partial n} = \frac{\partial y}{\partial s}$$

and, by the Cauchy-Riemann equations,

$$\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial s}$$

Thus, writting

$$\frac{\partial \psi}{\partial s} = U\frac{\partial y}{\partial s}; \qquad y>0$$

$$\frac{\partial \psi}{\partial s} = -U\frac{\partial y}{\partial s}; \quad y<0$$

we obtain the boundary condition (28) on the double section.

Appendix 2

Evaluation of the integral (30):

$$T=-\frac{\rho}{2}\int_0^{2\pi}\phi\frac{\partial\psi}{\partial\theta}d\theta \tag{30}$$

From Equation (24), over the unit circle

$$\phi=\frac{1}{2}\sum_{n=1}^{\infty}(b_ne^{-in\theta}+\bar{b}_ne^{in\theta}) \tag{a}$$

$$\frac{\partial\psi}{\partial\theta}=-\frac{1}{2}\sum_{n=1}^{\infty}n\,(b_ne^{-in\theta}+\bar{b}_ne^{in\theta}) \tag{b}$$

and with

$$\left.\begin{array}{l} b_n=\alpha_n+i\beta_n,\ \bar{b}_n=\alpha_n-i\beta_n \\ e^{\pm in\theta}=\cos n\theta\pm i\sin n\theta \end{array}\right\} \tag{c}$$

Equation (30) becomes

$$T=\frac{\rho}{8}\int_0^{2\pi}\sum_{n=1}^{\infty}(2\alpha_n\cos n\theta+2\beta_n\sin n\theta)\sum_{n=1}^{\infty}n\,(2\alpha_n\cos n\theta+2\beta_n\sin n\theta)\,d\theta \tag{d}$$

Integrating term by term, we obtain

$$T=\frac{\pi\rho}{2}\sum_{n=1}^{\infty}n\,|b_n|^2 \tag{e}$$

Finally, since $T'=\frac{1}{2}T$, we have

$$T'=\frac{\pi\rho}{4}\sum_{n=1}^{\infty}n\,|b_n|^2 \tag{31}$$

Appendix 3

Evaluation of Equations (48) and (49):

$$\sum_{n=1}^{\infty}(2n)\left(\frac{a_1-1}{1-4n^2}+\frac{ma_m}{m^2-4n^2}\right)^2; \qquad m=7 \text{ or } 11$$

$$=\sum_{n=1}^{\infty}\left\{\frac{2n}{(4n^2-1)^2}(1-a_1)^2+2\frac{-2nm}{(4n^2-1)(4n^2-m^2)}(1-a_1)a_m+\frac{2nm^2a_m^2}{(4n^2-m^2)^2}\right\}$$

$$=\alpha_{11}(1-a_1)^2+2\alpha_{1m}(1-a_1)a_m+\alpha_{mm}a_m^2$$

where

$$\alpha_{11}=\sum_{n=1}^{\infty}\frac{2n}{(4n^2-1)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right\}$$

$$=\frac{1}{4}\left\{\sum_{n=1}^{1}\frac{1}{(2n-1)^2}+\sum_{n=2}^{\infty}\frac{1}{(2n-1)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\right\}$$

$$=\frac{1}{4}\left\{1+\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\right\}=0.25$$

$$\alpha_{mm}=\sum_{n=1}^{\infty}\frac{2nm^2}{(4n^2-m^2)^2}=\frac{m}{4}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-m)^2}-\frac{1}{(2n+m)^2}\right\}$$

$$=\frac{m}{4}\left\{\sum_{n=1}^{m}\frac{1}{(2n-m)^2}+\sum_{n=m+1}^{\infty}\frac{1}{(2n-m)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n+m)^2}\right\}$$

$$=\frac{m}{4}\left\{\sum_{n=1}^{m}\frac{1}{(2n-m)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n+m)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n+m)^2}\right\}$$

$$=\frac{m}{4}\sum_{n=1}^{m}\frac{1}{(2n-m)^2}$$

hence,

$$\alpha_{7\cdot7}=\frac{7}{4}\left(\frac{1}{25}+\frac{1}{9}+1+1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}\right)=4.0646025\cdots$$

$$\alpha_{11\cdot11}=\frac{11}{4}\left(\frac{1}{81}+\frac{1}{49}+\frac{1}{25}+\frac{1}{9}+1+1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\frac{1}{121}\right)=6.5339845\cdots$$

and

$$\alpha_{1\cdot m}=\sum_{n=1}^{\infty}\frac{-2nm}{(4n^2-1)(4n^2-m^2)}=\frac{-m}{2(m-1)}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n+1)(2n-m)}-\frac{1}{(2n-1)(2n+m)}\right\}$$

$$=\frac{-m}{2(m-1)}\left\{\sum_{n=1}^{\frac{1}{2}(m-1)}\frac{1}{(2n+1)(2n-m)}+\sum_{n=\frac{1}{2}(m+1)}^{\infty}\frac{1}{(2n+1)(2n-m)}-\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+m)}\right\}$$

$$=\frac{-m}{2(m-1)}\left\{\sum_{n=1}^{\frac{1}{2}(m-1)}\frac{1}{(2n+1)(2n-m)}+\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+m)}-\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+m)}\right\}$$

$$=\frac{-m}{2(m-1)}\sum_{n=1}^{\frac{1}{2}(m-1)}\frac{1}{(2n+1)(2n-m)}$$

hence,

$$\alpha_{1\cdot7}=\frac{-7}{2(7-1)}\sum_{n=1}^{3}\frac{1}{(2n+1)(2n-7)}=\frac{7}{12}\sum_{n=1}^{3}\frac{1}{(2n+1)(7-2n)}$$

$$=\frac{7}{12}\left(\frac{1}{3\cdot5}+\frac{1}{5\cdot3}+\frac{1}{7\cdot1}\right)=0.16111109\cdots$$

$$\alpha_{1\cdot11}=\frac{-11}{2(11-1)}\sum_{n=1}^{5}\frac{1}{(2n+1)(2n-11)}=\frac{11}{20}\sum_{n=1}^{5}\frac{1}{(2n+1)(11-2n)}$$

$$=\frac{11}{20}\left(\frac{1}{3\cdot9}+\frac{1}{5\cdot7}+\frac{1}{7\cdot5}+\frac{1}{9\cdot3}+\frac{1}{11\cdot1}\right)=0.1221693\cdots$$

Appendix 4

From Equations (13) and (17), a_1 and a_m can be expressed in terms of p and σ of the section:

$$\left.\begin{aligned}a_1&=\frac{\gamma}{\kappa+m\pi}\left(m\pi+\sqrt{m\pi^2-(m-1)\pi\kappa}\right)\\a_m&=\frac{1}{\kappa+m\pi}\left(-\kappa+\sqrt{m\pi^2-(m-1)\pi\kappa}\right)\end{aligned}\right\};\ m=7\text{ or }11 \tag{57}$$

(Continued in page 24)

Appendix 5

Table 3. Numerical Results: $z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_7\zeta^{-7})$

p											
	a_m	0	0.01	0.02	0.03	0.04	0.04347				
	$-a_1$	0.6667	0.6733	0.68	0.6867	0.6933	0.6957				
0.2	σ	0.7854	0.7566	0.7267	0.6958	0.6641	0.6529				
	C_V	1.0	0.9476	0.9100	0.8863	0.8758	0.8750				
	C_H	0.4053	0.4038	0.4028	0.4022	0.4021	0.4022				
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.05263			
	$-a_1$	0.6	0.606	0.612	0.618	0.624	0.63	0.6316			
0.25	σ	0.7854	0.7604	0.7344	0.7077	0.6801	0.6518	0.6443			
	C_V	1.0	0.9554	0.9212	0.8968	0.8817	0.8753	0.8750			
	C_H	0.4053	0.4038	0.4027	0.4022	0.4021	0.4025	0.4026			
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.05882			
	$-a_1$	0.5556	0.5611	0.5667	0.5722	0.5778	0.5833	0.5882			
2/7	σ	0.7854	0.7622	0.7382	0.7135	0.6879	0.6618	0.6382			
	C_V	1.0	0.9594	0.9273	0.9033	0.8868	0.8775	0.8750			
	C_H	0.4053	0.4037	0.4027	0.4022	0.4021	0.4025	0.4033			
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.06666		
	$-a_1$	0.5	0.505	0.51	0.515	0.52	0.525	0.53	0.5333		
1/3	σ	0.7854	0.7640	0.7419	0.7191	0.6956	0.6714	0.6467	0.6300		
	C_V	1.0	0.9635	0.9339	0.9106	0.8935	0.8821	0.8761	0.8750		
	C_H	0.4053	0.4037	0.4026	0.4021	0.4022	0.4027	0.4036	0.4045		
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07692		
	$-a_1$	0.4286	0.4329	0.4371	0.4414	0.4457	0.45	0.4543	0.4615		
0.40	ρ	0.7854	0.7658	0.7455	0.7245	0.7029	0.6807	0.6580	0.6185		
	C_V	1.0	0.9678	0.9408	0.9189	0.9016	0.8889	0.8804	0.8750		
	C_H	0.4053	0.4036	0.4026	0.4021	0.4020	0.4029	0.4040	0.4078		
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.0909	
	$-a_1$	0.3333	0.3367	0.34	0.3433	0.3467	0.35	0.3533	0.36	0.3636	
0.50	σ	0.7854	0.7674	0.7487	0.7294	0.7096	0.6892	0.6684	0.6254	0.6014	
	C_V	1.0	0.9721	0.9481	0.9279	0.9112	0.8980	0.8879	0.8765	0.8750	
	C_H	0.4053	0.4035	0.4025	0.4021	0.4024	0.4033	0.4047	0.4093	0.4126	
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.10	0.11111
	$-a_1$	0.2	0.202	0.204	0.206	0.208	0.21	0.212	0.216	0.22	0.2222
2/3	σ	0.7854	0.7687	0.7514	0.7336	0.7152	0.6964	0.6771	0.6373	0.5961	0.5727
	C_V	1.0	0.9765	0.9558	0.9378	0.9223	0.9093	0.8985	0.8834	0.8760	0.8750
	C_H	0.4053	0.4034	0.4023	0.4021	0.4027	0.4041	0.4061	0.4122	0.4206	0.4262
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.10	0.12495
	$-a_1$	0.1111	0.1122	0.1133	0.1144	0.1156	0.1167	0.1178	0.12	0.1222	0.1250
0.80	σ	0.7854	0.7692	0.7524	0.7350	0.7172	0.6988	0.6801	0.6414	0.6014	0.5499
	C_V	1.0	0.9787	0.9598	0.9431	0.9284	0.9158	0.9051	0.8889	0.8791	0.8750
	C_H	0.4053	0.4032	0.4022	0.4022	0.4031	0.4048	0.4074	0.4148	0.4251	0.4412

p												
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.10	0.12	0.14285
	a_1	0	0	0	0	0	0	0	0	0	0	0
1.00	σ	0.7854	0.7694	0.7528	0.7357	0.7180	0.6999	0.6814	0.6432	0.6037	0.5630	0.5154
	C_V	1.0	0.9810	0.9639	0.9485	0.9349	0.9229	0.9124	0.8958	0.8843	0.8776	0.8750
	C_H	0.4053	0.4031	0.4021	0.4024	0.4038	0.4063	0.4097	0.4195	0.4326	0.4487	0.4704
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.10	0.12495	
	a_1	0.1111	0.1122	0.1133	0.1144	0.1156	0.1167	0.1178	0.12	0.1222	0.1250	
1.25	σ	0.7854	0.7692	0.7524	0.7350	0.7172	0.6988	0.6801	0.6414	0.6014	0.5499	
	C_V	1.0	0.9828	0.9672	0.9531	0.9404	0.9290	0.9189	0.9022	0.8899	0.8800	
	C_H	0.4053	0.4029	0.4021	0.4028	0.4049	0.4084	0.4132	0.4262	0.4436	0.4704	
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.10	0.11111	
	a_1	0.2	0.202	0.204	0.206	0.208	0.21	0.212	0.216	0.22	0.2222	
1.50	σ	0.7854	0.7687	0.7514	0.7336	0.7152	0.6964	0.6771	0.6373	0.5961	0.5727	
	C_V	1.0	0.9840	0.9695	0.9562	0.9441	0.9332	0.9235	0.9070	0.8944	0.8889	
	C_H	0.4053	0.4027	0.4021	0.4034	0.4064	0.4111	0.4173	0.4341	0.4562	0.4704	
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.0909		
	a_1	0.3333	0.3367	0.34	0.3433	0.3467	0.35	0.3533	0.36	0.3636		
2.00	σ	0.7854	0.7674	0.7487	0.7294	0.7096	0.6892	0.6684	0.6254	0.6014		
	C_V	1.0	0.9856	0.9723	0.9601	0.9490	0.9388	0.9295	0.9136	0.9063		
	C_H	0.4053	0.4025	0.4024	0.4051	0.4102	0.4178	0.4275	0.4532	0.4704		
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07692			
	a_1	0.4286	0.4329	0.4371	0.4414	0.4457	0.45	0.4543	0.4615			
2.50	σ	0.7854	0.7658	0.7455	0.7245	0.7029	0.6807	0.6580	0.6185			
	C_V	1.0	0.9865	0.9741	0.9626	0.9520	0.9422	0.9333	0.9200			
	C_H	0.4053	0.4023	0.4030	0.4074	0.4152	0.4261	0.4402	0.4704			
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.06666			
	a_1	0.5	0.505	0.51	0.515	0.52	0.525	0.53	0.5333			
3.00	σ	0.7854	0.7640	0.7419	0.7191	0.6956	0.6714	0.6467	0.6300			
	C_V	1.0	0.9872	0.9752	0.9642	0.9540	0.9446	0.9359	0.9306			
	C_H	0.4053	0.4020	0.4040	0.4105	0.4214	0.4366	0.4556	0.4704			
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.05882				
	a_1	0.5556	0.5611	0.5667	0.5722	0.5778	0.5833	0.5882				
3.50	σ	0.7854	0.7622	0.7382	0.7135	0.6879	0.6618	0.6382				
	C_V	1.0	0.9876	0.9761	0.9654	0.9554	0.9463	0.9388				
	C_H	0.4053	0.4021	0.4051	0.4143	0.4289	0.4487	0.4704				
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.05263				
	a_1	0.6	0.606	0.612	0.618	0.624	0.63	0.6316				
4.00	σ	0.7854	0.7604	0.7344	0.7077	0.6801	0.6518	0.6443				
	C_V	1.0	0.9879	0.9767	0.9662	0.9566	0.9476	0.9453				
	C_H	0.4053	0.4023	0.4068	0.4186	0.4375	0.4627	0.4704				
	a_m	0	0.01	0.02	0.03	0.04	0.04347					
	a_1	0.6667	0.6733	0.68	0.6867	0.6933	0.6957					
5.00	σ	0.7854	0.7566	0.7267	0.6958	0.6641	0.6529					
	C_V	1.0	0.9884	0.9776	0.9675	0.9581	0.9550					
	C_H	0.4053	0.4025	0.4108	0.4295	0.4582	0.4704					

Table 4. Numberical Results: $z(\zeta)=R(\zeta+a_1\zeta^{-1}+a_{11}\zeta^{-11})$

p	a_m	0	0.01	0.02	0.02857					
	$-a_1$	0.6667	0.6733	0.68	0.68571					
0.20	σ	0.7854	0.7560	0.7245	0.6959					
	C_V	1.0	0.9511	0.9239	0.9167					
	C_H	0.4053	0.4032	0.4019	0.4013					
	a_m	0	0.01	0.02	0.03	0.03448				
	$-a_1$	0.60	0.606	0.612	0.618	0.62069				
0.25	σ	0.7854	0.7599	0.7326	0.7035	0.6899				
	C_V	1.0	0.9579	0.9308	0.9180	0.9168				
	C_H	0.4053	0.4031	0.4018	0.4013	0.4013				
	a_m	0	0.01	0.02	0.03	0.03846				
	$-a_1$	0.5556	0.5611	0.5667	0.5722	0.5769				
2/7	σ	0.7854	0.7618	0.7365	0.7096	0.6857				
	C_V	1.0	0.9614	0.9351	0.9205	0.9167				
	C_H	0.4053	0.4031	0.4017	0.4012	0.4014				
	a_m	0	0.01	0.02	0.03	0.04	0.04347			
	$-a_1$	0.50	0.505	0.51	0.515	0.52	0.5217			
1/3	σ	0.7854	0.7636	0.7403	0.7155	0.6894	0.6800			
	C_V	1.0	0.9651	0.9400	0.9242	0.9172	0.9167			
	C_H	0.4053	0.4030	0.4017	0.4012	0.4016	0.4020			
	a_m	0	0.01	0.02	0.03	0.04	0.05			
	$-a_1$	0.4286	0.4329	0.4371	0.4414	0.4457	0.4500			
0.40	σ	0.7854	0.7654	0.7440	0.7212	0.6972	0.6720			
	C_V	1.0	0.9690	0.9455	0.9293	0.9198	0.9167			
	C_H	0.4053	0.4029	0.4016	0.4013	0.4017	0.4033			
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.05882		
	$-a_1$	0.3333	0.3367	0.34	0.3433	0.3467	0.35	0.3529		
0.50	σ	0.7854	0.7670	0.7474	0.7264	0.7044	0.6812	0.6600		
	C_V	1.0	0.9730	0.9516	0.9355	0.9246	0.9184	0.9167		
	C_H	0.4053	0.4028	0.4015	0.4013	0.4022	0.4040	0.4064		
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.07142
	$-a_1$	0.2	0.202	0.204	0.206	0.208	0.21	0.212	0.214	0.2143
2/3	σ	0.7854	0.7684	0.7502	0.7308	0.7104	0.6889	0.6666	0.6433	0.6400
	C_V	1.0	0.9771	0.9582	0.9431	0.9316	0.9235	0.9186	0.9167	0.9167
	C_H	0.4053	0.4026	0.4014	0.4015	0.4029	0.4056	0.4093	0.4142	0.4150
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
	$-a_1$	0.1111	0.1122	0.1133	0.1144	0.1156	0.1167	0.1178	0.1189	0.12
0.80	σ	0.7854	0.7689	0.7512	0.7323	0.7125	0.6916	0.6699	0.6473	0.6240
	C_V	1.0	0.9792	0.9617	0.9474	0.9359	0.9273	0.9213	0.9178	0.9167
	C_H	0.4053	0.4024	0.4013	0.4017	0.4037	0.4070	0.4117	0.4176	0.4248

p											
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.0909
	a_1	0	0	0	0	0	0	0	0	0	0
1.00	σ	0.7854	0.7691	0.7516	0.7330	0.7134	0.6928	0.6713	0.6490	0.6260	0.6000
	C_V	1.0	0.9814	0.9654	0.9519	0.9408	0.9320	0.9252	0.9205	0.9177	0.9166
	C_H	0.4053	0.4022	0.4012	0.4022	0.4050	0.4096	0.4158	0.4236	0.4328	0.4443
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	
	a_1	0.1111	0.1122	0.1133	0.1144	0.1156	0.1167	0.1178	0.1189	0.12	
1.25	σ	0.7854	0.7689	0.7512	0.7323	0.7125	0.6916	0.6699	0.6473	0.6240	
	C_V	1.0	0.9831	0.9684	0.9558	0.9451	0.9363	0.9293	0.9238	0.9200	
	C_H	0.4053	0.4020	0.4013	0.4031	0.4072	0.4135	0.4219	0.4322	0.4444	
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.07142	
	a_1	0.20	0.202	0.204	0.206	0.208	0.21	0.212	0.214	0.2143	
1.50	σ	0.7854	0.7684	0.7502	0.7308	0.7104	0.6889	0.6666	0.6433	0.6400	
	C_V	1.0	0.9843	0.9705	0.9585	0.9482	0.9395	0.9324	0.9266	0.9259	
	C_H	0.4053	0.4018	0.4015	0.4043	0.4099	0.4182	0.4290	0.4423	0.4444	
	a_m	0	0.01	0.02	0.03	0.04	0.05	0.05882			
	a_1	0.3333	0.3367	0.34	0.3433	0.3467	0.35	0.3529			
2.00	σ	0.7854	0.7670	0.7474	0.7264	0.7044	0.6812	0.6600			
	C_V	1.0	0.9858	0.9732	0.9620	0.9523	0.9439	0.9375			
	C_H	0.4053	0.4015	0.4023	0.4074	0.4167	0.4298	0.4444			
	a_m	0	0.01	0.02	0.03	0.04	0.05				
	a_1	0.4286	0.4329	0.4371	0.4414	0.4457	0.45				
2.50	σ	0.7854	0.7654	0.7440	0.7212	0.6972	0.6720				
	C_V	1.0	0.9867	0.9748	0.9642	0.9549	0.9467				
	C_H	0.4053	0.4012	0.4035	0.4117	0.4254	0.4444				
	a_m	0	0.01	0.02	0.03	0.04	0.04347				
	a_1	0.50	0.505	0.5	0.515	0.52	0.5217				
3.00	σ	0.7854	0.7636	0.7403	0.7155	0.6894	0.6800				
	C_V	1.0	0.9873	0.9759	0.9657	0.9566	0.9537				
	C_H	0.4053	0.4013	0.4054	0.4172	0.4361	0.4444				
	a_m	0	0.01	0.02	0.03	0.03846					
	a_1	0.5556	0.5611	0.5667	0.5722	0.5769					
3.50	σ	0.7854	0.7618	0.7365	0.7096	0.6857					
	C_V	1.0	0.9878	0.9767	0.9668	0.9592					
	C_H	0.4053	0.4013	0.4076	0.4237	0.4444					
	a_m	0	0.01	0.02	0.03	0.03448					
	a_1	0.60	0.606	0.612	0.618	0.62069					
4.00	σ	0.7854	0.7599	0.7326	0.7035	0.6899					
	C_V	1.0	0.9881	0.9773	0.9676	0.9536					
	C_H	0.4053	0.4016	0.4105	0.4313	0.4444					
	a_m	0	0.01	0.02	0.02857						
	a_1	0.6667	0.6733	0.68	0.68571						
5.00	σ	0.7854	0.7560	0.7245	0.6959						
	C_V	1.0	0.9885	0.9781	0.9700						
	C_H	0.4053	0.4023	0.4175	0.4444						

(Continued from page 19)

where

$$\gamma=\frac{p-1}{p+1} \quad \text{(a)}$$

$$\kappa=(\pi-4\sigma)\gamma^2+4\sigma \quad \text{(b)}$$

Hence, we can calculate C_V and C_H directly by employing p and σ, that is, by substituting Equation (57) for a_1 and a_m of the following equations which are consistent with Equations (40) and (51) for $m=7$, or (42) and (53) for $m=11$:

$$C_V=\frac{(1+2a_1+a_1{}^2+ma^2{}_m)}{(1+a_1+a_m)^2}; \qquad m=7 \text{ or } 11 \quad \text{(c)}$$

$$C_H=\frac{16}{\pi^2}\frac{1}{(1-a_1+a_m)^2}\{\alpha_{11}(1-a_1)+2\alpha_{1m}(1-a_1)a_m+\alpha_{mm}\,a_m{}^2\}; \qquad m=7 \text{ or } 11 \quad \text{(d)}$$

Here, it is noted again that the values of σ for each p must be within the range defined by Equations (20) and (21), or given in Table 1.