Some Remarks on Non-Symmetric Affine Connections

OK KYUNG YOON

We consider an n-dimensional compact space $V_n (n \geqq 3)$ on which there is given a positive definite metric

(1) $$ds^2 = g_{jk} dx^j dx^k$$

and a general affine connection $E^i{}_{jk}$. Thus we may have

(2) $$E^i{}_{jk} \neq E^i{}_{kj}.$$

Covariant differentiation with respect to it will be donoted by a solidus, so that

(3) $$g_{jk|l} = \frac{\partial g_{jk}}{\partial x^l} - g_{sk} E^s{}_{jl} - g_{js} E^s{}_{kl}.$$

We will also introduce its curvature tensor

(4) $$E^i{}_{jkl} = \frac{\partial E^i{}_{jk}}{\partial x^l} - \frac{\partial E^i{}_{jl}}{\partial x^k} + E^h{}_{jk} E^i{}_{hl} - E^h{}_{jl} E^i{}_{hk}.$$

Furthermore, we assume that the covariant derivative of the metric tensor g_{jk} is proportional to the metric tensor, so that

(5) $$g_{jk|l} = 2\sigma_l g_{jk}$$

where σ_l is a covariant vector. If we solve (5) with respect to $E^i{}_{jk}$ we obtain

(6) $$E^i{}_{jk} = \left\{ {i \atop jk} \right\} - \delta^i{}_k \sigma_j + T^i{}_{jk}$$

where we have put

(7) $$\left\{ {i \atop jk} \right\} = \frac{1}{2} g^{ih} \left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right)$$

and

(8) $$g_{li} T^l{}_{jk} + g_{jl} T^l{}_{ik} \equiv T_{ijk} + T_{jik} = 0.$$

If we define the torsion tensor

(9) $$S_{jk}{}^i = \frac{1}{2} (E^i{}_{jk} - E^i{}_{kj})$$

then we obtain

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(10) $$E^i_{jk} = \left\{ {i \atop jk} \right\} - \delta^i_k \sigma_j - \delta^i_j \sigma_k + \sigma^i g_{jk} + S_{jk}{}^i - S^i{}_{jk} - S^i{}_{kj}$$

where $S^i{}_{jk} = g^{ih} g_{kt} S_{hj}{}^t$ and $\sigma^i = \sigma_t g^{ti}$. On putting

(11) $$\Omega_{jk}{}^i = S_{jk}{}^i - S^i{}_{jk} - S^i{}_{kj}$$

we find

(12) $$E^i{}_{jkl} = R^i{}_{jkl} + \delta^i_j(\sigma_{lk} - \sigma_{kl}) + \delta^i_l \sigma_{jk} - \delta^i_k \sigma_{jl} + \sigma^i{}_l g_{jk} - \sigma^i{}_k g_{jl} + \sigma^i(\Omega_{jkl} - \Omega_{jlk}) + \sigma_j(\Omega_{lk}{}^i - \Omega_{kl}{}^i) + \Omega^i{}_{jkl}$$

where, $R^i{}_{jkl}$ is the Riemannian cuvrature tensor for $\left\{ {i \atop jk} \right\}$, and

$$\sigma_{jk} = \sigma_{j,k} + \sigma_j \sigma_k - \frac{1}{2} \sigma_h \sigma_{,i} g^{hm} g_{jk} - \sigma_m \Omega_{jk}{}^m$$

$$\sigma^i{}_k = \sigma_{jk} g^{ik}$$

$$\Omega^i{}_{jkl} = \Omega_{jk}{}^i{}_{,l} - \Omega_{jl}{}^i{}_{,k} + \Omega_{jk}{}^m \Omega_{ml}{}^i - \Omega_{jl}{}^m \Omega_{mk}{}^i$$

and the comma denotes covariant differentiation with respect to the Christoffel symbols introduced. From (12), on putting $E_{ijkl} = g_{im} E^m{}_{jkl}$ and $\Omega_{ijkl} = g_{im} \Omega^m{}_{jkl}$, we obtain

(13) $$E_{ijkl} + E_{jikl} = 2g_{ij}(\sigma_{l,k} - \sigma_{k,l}) + \Omega_{ijkl} + \Omega_{jikl} \quad \text{and} \quad E_{ijkl} + E_{ijlk} = 0.$$

As for the Ricci contraction, the customary concept is

(14) $$E_{jk} = E^m{}_{jkm},$$

but we will form it completely differently, namely as

(15) $$E^i{}_j = \frac{1}{2} g^{kl} (E^i{}_{klj} + E^i{}_{lkj})$$

and we will then form the contraction

(16) $$E = g^{ij} E_{ij}$$

will be called the pseudo-Ricci tensor, and the pseudo-scalar curvature.

1. σ-transfomation

We assume that $T^i{}_{jk} = 0$. In this case we obtain

(1-1) $$E^i{}_{jk} = \left\{ {i \atop jk} \right\} - \delta^i_j \sigma_k,$$

$$E^i{}_{jkl} = R^i{}_{jkl} + \delta^i_j(\sigma_{l,k} - \sigma_{k,l}) \quad \text{and} \quad E_{jk} = R_{jk} + (\sigma_{j,k} - \sigma_{k,j}).$$

If we take $\sigma_{l,k} = \sigma_{k,l}$ then $E^i{}_{jkl} = R^i{}_{jkl}$ and $E_{jk} = R_{jk}$. So that the curvature is invariant from σ-transformation.

Next, we introduce the path of connection $E^i{}_{jk}$ in V_n by

$$\frac{d^2x^i}{dt^2}+E^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0. \tag{1-2}$$

If we put, the solution of two equations

$$\frac{d^2x^i}{ds^2}+\left\{ {i \atop jk} \right\}\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \qquad ds^2=g_{ij}\,dx^i\,dx^j$$

is $x^i=f^i(s)$ that is geodesic of V_n, and

$$\phi=\sigma_k(f^1(s)f^2(s),\cdots\cdots,f^n(s))\frac{df^k(s)}{ds}.$$

We obtain, the solution of (1-2) is

$$X^i=f^i(\varphi(t))$$

where $s=\varphi(t)$ is the solution of $c_1t+c_2=\int e^{-\int\phi ds}\,ds$.

Therefore we obtain the following theorem.

THEOREM 1. *In a V_n, if $\sigma_{l,k}=\sigma_{k,l}$, then the curvature and path are invariant from σ-transformation* $E^i{}_{jk}=\left\{ {i \atop jk} \right\}-\sigma^i{}_j\,\sigma_k$.

2. A space with no torsion

If we assume that $S_{ijk}=0$, then we have $\Omega_{jk}{}^i=0$ and (12) becomes

$$E^i{}_{jkl}=R^i{}_{jkl}+\delta^i{}_j(\sigma_{lk}-\sigma_{kl})+\delta^i{}_l\,\sigma_{jk}-\delta^i{}_k\,\sigma_{jl}+g_{jk}\,\sigma^i{}_l-g_{jl}\,\sigma^i{}_k \tag{2-1}$$

we also have

$$E_{jk}=R_{jk}+(\sigma_{jk}-\sigma_{kj})+(n-2)\sigma_{jk}+\sigma^l{}_l\,g_{jk} \tag{2-2}$$

and

$$E=R+2(n-1)\sigma^l{}_l. \tag{2-3}$$

From (2-2) we deduce

$$\frac{1}{2}(E_{jk}+E_{kj})=R_{jk}+\frac{n-1}{2}(\sigma_{jk}+\sigma_{kj})+\sigma^l{}_l\,g_{jk} \tag{2-4}$$

$$\frac{1}{2}(E_{jk}-E_{kj})=n(\sigma_{jk}-\sigma_{kj}) \tag{2-5}$$

and thus the tensor E_{jk} is not symmetric in general, although $E^i{}_{jk}=E^i{}_{kj}$. Eliminating σ_{jk}, $\sigma^l{}_l$ from (2-1), (2-2) and (2-3), the resulting equations are reducible to

$$\bar{E}^i{}_{jkl}=\bar{R}^i{}_{jkl} \tag{2-6}$$

where

$$\overline{E}^i{}_{jkl} = E^i{}_{jkl} + \frac{1}{n}\delta^i{}_j(E_{kl}-E_{lk}) \tag{2-7}$$
$$+\frac{1}{n(n-2)}\{\delta^i{}_k(E_{lj}-E_{jl})-\delta^i{}_l(E_{kj}-E_{jk})+g_{jl}(E_k{}^i-E^i{}_k)-g^i{}_k(E^i{}_l-E^i{}_l)\}$$
$$+\frac{1}{n-2}(\delta^i{}_k E_{jl}-\delta^i{}_l E_{jk}+g_{jl}E^i{}_k-g_{jk}E^i{}_l)$$
$$+\frac{E}{(n-1)(n-2)}(\delta^i{}_l g_{jk}-\delta^i{}_k g_{jl})$$

The tensor $\overline{E}^i{}_{jkl}$ is invariant from covariant vector σ_l, This tensor will be called pseudoconformal curvature tensor.

If V_n is conformally flat space, then $\overline{E}^i{}_{jkl}=0$. In this case we call that V_n is pseudo-conformally flat space.

It will be easily verified that the components $E^i{}_{jkl}$ and $\overline{E}^i{}_{jkl}$ of the curvature tensors satisfy, instead of the usual ones the following first Bianchi identities

$$\begin{aligned} E^i{}_{jkl}+E^i{}_{klj}+E^i{}_{ljk}&=0 \\ \overline{E}^i{}_{jkl}+\overline{E}^i{}_{klj}+\overline{E}^i{}_{ljk}&=0 \end{aligned} \tag{2-8}$$

If the vector σ_i is a killing vector, that is

$$\sigma_{i\,,\,j}+\sigma_{j\,,\,i}=0 \tag{2-9}$$

then

$$\frac{1}{2}(E_{jk}+E_{kj}) = R_{jk}+(n-2)(\sigma_j\sigma_k-\sigma^l\sigma_l g_{jk})$$

and then we have

$$E_{jk}\xi^j\xi^k = R_{jk}\xi^j\xi^k-(n-2)\{(\sigma^k\sigma_l)(\xi^m\xi_m)-(\sigma_m\xi^m)^2\} \tag{2-10}$$

Hence, we obtain the following theorems.

THEOREM 2. *In V_n with symmetric connection tensor $E^i{}_{jk}$, if $\sigma_{i\,,\,j}+\sigma_{j\,,\,i}=0$ and $E_{jk}+E_{kj}=0$, then $R_{jk}\xi^j\xi^k$ is non-negative.*

THEOREM 3. *In V_n with symmetric connection tensor $E^i{}_{jk}$, if $\sigma_{i\,,\,j}+\sigma_{j\,,\,i}=0$ and $R_{jk}\xi^j\xi^k$ is non-positive, then $E_{jk}\xi^j\xi^k$ is non-positive.*

These two theorems were derived by Bochner and Yano, by replacing antisymmetric torsion tensor ([2]).

Next, if the vector σ_i is a pseudo-killing vector, that is

$$\sigma_{i\,|\,j}+\sigma_{j\,|\,i}=0 \tag{2-11}$$

then

$$\frac{1}{2}(E_{jk}+E_{kj}) = R_{jk}+(n-2)(\sigma^l \sigma_l g_{jk}-\sigma_j \sigma_k)$$

and then we have

$$E_{jk}\xi^j\xi^k=R_{jk}\xi^j\xi^k+(n-2)\{(\sigma^l\sigma_l)(\xi^m\xi_m)-(\sigma_m\xi^m)^2\}. \tag{2-21}$$

Hence, we obtain the following theorems.

THEOREM 4. *In V_n with symmetric connection tensor E^i_{jk} if $\sigma_{i|j}+\sigma_{j|i}=0$ and $E_{jk}+E_{kj}=0$, then $R_{jk}\xi^j\xi^k$ is non-positive.*

THEOREM 5. *In V_n with symmetric connection tensor E_{jk} if $\sigma_{i|j}+\sigma_{j|i}=0$ and $R_{jk}\xi^j\xi^k$ is non-negative, then $E_{jk}\xi^j\xi^k$ is non-negative.*

References

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Seoul National University