On the Identity Component of Topological Groups

JAIHAN YOON

1. Let G be a locally compact topological group. Although the identity component G_0 of G, in general, is not open, some algebraic condition imposed on G_0 may happen to ensure that G_0 to be open. One of these conditions will be found in Theorem 1, and it will be shown that such groups are σ-compact.

In order to prove the σ-compactness, we need the following known theorem;

A locally compact space is σ-compact if and only if every open subgroup of G is of countable index.

It is well known that every locally compact group has an open subgroup G' such that G'/G_0 is approximated by Lie groups. The group G' is merely any open subgroup of G such that G'/G_0 is compact [1].

The existence of open normal subgroup of G that can be approximated by Lie groups will be shown in Theorem 2, where G has small invariant neighborhoods of the identity.

Groups with small invariant neighborhoods were first studied by G. Mostow [2].

2. THEOREM 1. *Let G be a locally compact group whose identity component G_0 is of countable index. Then the component G_0 is open.*

Proof. From the fact that G_0 is closed and of countable index in the group G, G is the union of countable cosets g_iG_0, each of which is closed. Hence one of g_iG_0's contains an open set of G. Each of g_iG_0's being homeomorphic to G_0, G_0 contains an interior point, and this proves that G_0 is open.

COROLLARY. *G is σ-compact.*

Proof. It is well known that the identity component of a locally compact group is the intersection of all open subgroups. Now, it is easy to see that the index of any open subgroup of G does not exceed that of G_0. The σ-compactness of G follows from the theorem stated in § 1.

THEOREM 2. *Let G be a locally compact group with small invariant neighborhoods of the identity. Then G has an open normal subgroup of G which can be approximated by Lie groups.*

Proof. The quotient group $G_1 = G/G_0$ is totally disconnected and also has small invariant neighborhoods of the identity. Since G_1 is locally compact and totally disconnected, every neighborhood of the identity contains an open compact subgroup

Received by the editors September 22, 1968.

H. Let V_1 be an invariant neighborhood of the identity contained in H. Then, $H' = \bigcap_{x \in G_1} xHx^{-1}$ contains V_1, and is an open compact subgroup of G_1. It is clear that the inverse image N of H' under the canonical homomorphism

$$T : G \longrightarrow G_1 = G/G_0$$

is an open normal subgroup of G containing G_0. Now we have

$$N/G_0 = T^{-1}(H')/G_0 = H'.$$

Hence N/G_0 is compact, which proves the theorem.

References

1. D. Montgomery and L. Zippin, *Topological Transformation Groups*, Interscience, N.Y., 1955.
2. G.D. Mostow, *On an assertation of Weil*, Ann. of Math., 54(1951).

Seoul National University