Point-like Decompositions of S^n Which Do Not Yield S^n

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If H is a collection of disjoint compacta in a space, we denote by H^* the sum of members of H. An element g of H is said to be *central* or *noncentral* in H according as g contains a limit point of H^*-g or not. If α is an ordinal then, by letting $H=H_0$, we define inductively H_α to be the collection of members of H which are central in all H_β for $\beta<\alpha$. As in [2], an element of H is said to be *essential* or *inessential* in H according as it belongs to H_α for all countable ordinal α or not; H is a *countably discrete* collection if $H_\alpha=\phi$ for some countable ordinal α.

Let G be an upper semicontinuous decomposition of the n-sphere S^n, and let H denote the collection of nondegenerate elements of G. J. L. Bailey [1] announces that if H is a countable collection of point-like continua and the decomposition space of G is not homeomorphic with S^n then there is a subcollection H' of H in which every element is noncentral such that H' is the collection of nondegenerate elements of some upper semicontinuous decomposition G' of S^n whose decomposition space is not homeomorphic with S^n. The purpose of this note is to point out how the results of our earlier work [2] imply that this remains still true even if H is not required to be a countable collection. Note also that the other of the two theorems reported in [1] is a special case of Lemma 7 of [2].

LEMMA 1. *Let H be a collection of disjoint compacta in a separable metric space. If each element of H is inessential then H is a countably discrete collection.*

Proof. Let x be a point in $H^*-H_\alpha{}^*$, and let g be the unique element of H containing x. Then $g\in H_\beta-H_{\beta+1}$ for some $\beta<\alpha$, and there is a neighborhood of x disjoint from $H_\alpha{}^*$. Thus $H^*-H_\alpha{}^*$ is open in H^*, and the sets $H^*-H_\alpha{}^*$ form an open cover of H^*, where the α's run over the set of countable ordinals. Since a separable metric space is hereditarily Lindelöf, there are countable ordinals α_n, $n=1, 2, \cdots\cdots$, such that the sets $H^*-H_{\alpha_n}{}^*$ also cover H^*. On the other hand, a countable set cannot be cofinal in the set of countable ordinals, and so there is a countable ordinal α with $\alpha_n\leq\alpha$ for all n. For this α, we have $H^*-H_\alpha{}^*=H^*$, completing the proof.

LEMMA 2. *Let G be an upper semicontinuous decomposition of the n-sphere S^n, and let H and H' denote, respectively, the collection of nondegenerate elements of G and the collection of essential elements of H. Then the decomposition G' of S^n into points and members of H' is upper semicontinuous.*

Proof. By Lemma 1, the collection of inessential elements of H is countably discrete,

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and we have $H' = H_\alpha$ for some countable ordinal α. Upper semicontinuity of G' follows from [2, Lemma 4].

THEOREM. *Let G be an upper semicontinuous decomposition of the n-sphere S^n into point-like continua, and let H denote the collection of nondegenerate elements of G. If the decomposition space of G is not homeomorphic with S^n then there is a subcollection H' of H in which every element is essential such that H' is the collection of nondegenerate elements of some upper semicontinuous decomposition G' having the decomposition space homeomorphic with that of G.*

Proof. Let H' be the collection of essential elements of H. Then H' is nonempty as otherwise the decomposition space of G must be homeomorphic with S^n by Lemma 2 and [2, Theorem 2]. It is also obvious that each element of H' is essential. That this H' satisfies all claims of the theorem follows now from Lemma 7 of [2].

References

1. J. L. Bailey, *Point-like upper semicontinuous decompositions of S^3*, Notices Amer. Math. Soc., **14** (1967), 917.
2. Jehpill Kim, *A note on upper semicontinuous decompositions of the n-sphere*, Duke Math. J., **33** (1966), 683–687.

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