On a Conjecture of Fekete

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1. INTRODUCTION. We consider the polynomial

$$f(x)=\left(\frac{1}{p}\right)+\left(\frac{2}{p}\right)x+\left(\frac{3}{p}\right)x^2+\cdots\cdots+\left(\frac{p-1}{p}\right)x^{p-2}, \tag{1}$$

where p is an odd prime and $\left(\frac{n}{p}\right)$ Legendre's symbol. Once Fekete conjectured (see [4]) that $f(x)$ has no zero in $(0,\ 1)$. The significance of this conjecture lies in that if it were true, then from the equation(see [1])

$$\Gamma(s)L(s,\ \chi) = \int_0^1\left(\log\frac{1}{x}\right)^{s-1}\frac{f(x)}{1-x^p}\frac{dx}{x} \quad \text{for} \quad \operatorname{Re} s > 0,$$

where $L(s,\ \chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ with $\chi(n)=\left(\frac{n}{p}\right)$, we could conclude the positivity of $L(s,\ \chi)$ for positive s which is, still now, one of the most searched problem in the theory of numbers (see e.g. [5]).

In 1919, Polya [4] gave the following counter-example: for any prime p, satisfying the condition

$$\left(\frac{q}{p}\right) = \begin{cases} -1 & \text{for } q=2,\ 3,\ 5,\ 7,\ 11,\ 13, \\ +1 & \text{for other primes less than } p, \end{cases} \tag{2}$$

$f(x)$ becomes negative at $x=0.7$(note $f(0)>0$). In fact, there is an infinity of such primes p. He also stated that $[0,\ 0.5]$ is a zero-free domain for any Fekete polynomial (1).

We found, however, that (1) is negative around $x=0.75$ for only $q=2,\ 3,\ 5,\ 7$ and 23 for which $\left(\frac{q}{p}\right)=-1$ in (2). This means that $q=23$ can be replaced by any one of 11, 13, 17 and 19. We actually computed, using computer,

$$\sum_{n=1}^{42}\left(\frac{n}{p}\right)x^n+\frac{x^{43}}{1-x} \quad \text{(satisfying the above)}$$

which is definitely greater than (1) for all x in $(0, 1)$ (e.g. it equals -0.0012 at $x=0.745$). Also by computing this time

$$\sum_{n=1}^{42}\left(\frac{n}{p}\right)x^n-\frac{x^{43}}{1-x}$$

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for the worst case where $\left(\frac{q}{p}\right)=-1$ for all primes $q<42$, we extend a zero-free domain for every (1) to be $[0, 0.6739]$. Finally, we note that the existence of an infinitude of Fekete polynomials (1) which are positive for all x in $(0, 1)$ is still remained as an open question.

The purpose of this note is to give an extension of Polya's example in the sense stated as the following theorem.

2. THEOREM. *Given a finite sequence* $\{\varepsilon_i\}$ $(1\leqq i\leqq k)$ *of* $+1$ *or* -1 *such that*

$$\left(\frac{p_i}{p}\right)=\varepsilon_i, \text{ where } p_i \text{ is the } i\text{-th prime},$$

there exist infinitely many primes p *for which*

$$f(x)=\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)x^{n-1}<0$$

for some x *in* $(0, 1)$.

Proof. We apply here the method used by Heilbronn [2]. Let $P=\{p_i\,|\,\varepsilon_i=+1\}$. Since $\zeta(s)=\prod_p(1-p^{-s})^{-1}$ for $s>1$,

$$\begin{aligned}
&\left(\prod_{p\in P}\frac{1+p^{-2s}}{1-p^{-s}}\right)\frac{\zeta(2s)}{\zeta(s)}\\
(3)\qquad &=\prod_{p\in P}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\cdots\right)\cdot\prod_{p\notin P}\left(1-\frac{1}{p^s}+\frac{1}{p^{2s}}-\frac{1}{p^{3s}}+\cdots\cdots\right)\\
&=\sum_{n=1}^{\infty}\frac{\lambda_P(n)}{n^s},
\end{aligned}$$

where $\lambda_P(n)=(-1)^\rho$ with $\rho=$ the number of prime factors of n, counting multiplicitly, except the primes in P. Note that since $\zeta(s)=\frac{1}{s-1}+0(1)$, (3) $\to 0$ as $s\to 1$. Now let

$$F(x)=\sum_{n=1}^{\infty}\lambda_P(n)e^{-nx} \quad \text{for } x>0.$$

Since

$$n^{-s}\Gamma(s)=n^{-s}\int_0^\infty y^{s-1}e^{-y}\,dy=\int_0^\infty\left(\frac{y}{n}\right)^{s-1}e^{-y}\frac{dy}{n}=\int_0^\infty x^{s-1}e^{-nx}\,dx,$$

$$\Gamma(s)\sum_{n=1}^{\infty}\frac{\lambda_P(n)}{n^s}=\sum_{n=1}^{\infty}\frac{\lambda_P(n)}{n^s}\int_0^\infty y^{s-1}e^{-y}\,dy$$

$$=\sum_{n=1}^{\infty}\lambda_P(n)\int_0^\infty x^{s-1}e^{-nx}\,dx=\lim_{n\to\infty}\int_0^\infty x^{s-1}\sum_{n=1}^{N}\lambda_P(n)e^{-nx}\,dx$$

$$=\int_0^\infty x^{s-1}F(x)\,dx.$$

Here the interchange between limit and integral is justified by absolute convergence. By the above note this integral tends to 0 as s tends to 1. Now $F(x)$, not being identically zero, $F(x_0)<0$ for some $x_0>0$. Then we can choose an integer $a>0$ such that

$$F(x_0)+2\sum_{n=a+1}^{\infty} e^{-nx_0}=F(x_0)+2\frac{e^{-ax_0}}{e^{x_0}-1}<0. \tag{4}$$

Also choose a prime $p(>a)$ such that

$$\left(\frac{p_i}{p}\right)=\varepsilon_i \quad \text{for } i=1,\ 2, \cdots\cdots, k$$

and

$$\left(\frac{p_i}{p}\right)=1 \quad \text{for } p_k<p_i\leqq a.$$

From [3] we know the existence of an infinitude of such primes p. Clearly

$$\left(\frac{n}{p}\right)=\lambda_p(n) \quad \text{for } n\leqq a.$$

Form (4)

$$\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)e^{-nx_0}\leqq F(x_0)+2\sum_{n=a+1}^{\infty} e^{-nx_0}<0.$$

Let $e^{-x_0}=y_0$. Then $0<y_0<1$ and

$$\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)y_0^{n}=\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)y_0^{n-1}\,(y_0+y_0^{p+1}+y_0^{2p+1}+\cdots\cdots)<0.$$

Therefore

$$f(y_0)=\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)y_0^{n-1}<0.$$

REMARK. In the hypothesis of Theorem one can assign each i to any one of $+1$, -1 and including 0. If $P_0=\{p_i\,|\,\varepsilon_i=0\}$, then we take

$$\chi(n)=\Lambda_p(n)=\left(\frac{n}{p}\right)\chi_0(n),$$

instead of

$$\chi(n)=\lambda_p(n)=\left(\frac{n}{p}\right)$$

where χ_0 is the principal character of module $\prod_{p\in P_0} p$, i.e.,

$$\chi_0(n)=\begin{cases}0 & \text{if } p\,|\,n \text{ for some } p\in P_0,\\ 1 & \text{otherwise.}\end{cases}$$

And thus for

$$L(s,\ \chi)=\Sigma\frac{\lambda_p(n)}{n^s}=\prod_p\frac{1}{1-\frac{\chi(p)}{p^s}}$$

we take

$$L(s,\ \chi)=\Sigma\frac{\Lambda_p(n)}{n^s}=\prod_{q\in P_0}\left(1-\frac{\chi(q)}{q^s}\right)L(s,\ \chi).$$

References

1. Gottfried Grimm, *Über die Reellen Nullstellen Dirichletscher Reihen,* Math. Zentralbratt, **5** (1932), p.16.
2. Hans Heilbronn, *On real characters,* Acta Arith. **2** (1936), 212-3.
3. David Hilbert, *Die Theorie der algebraischen Zahlkörper,* Jahr. der deut. Math. (1897), Satz III.
4. Georg Polya, *Über eine Vermutung des Herrn Fekete,* Deut. Math. Vereinigung, 28 (1919), p.37.
5. J. Barkley Rosser, *Real Dirichlet L-series,* J. of Research of the National Bureau of Standards, Vol. 45, No. 6 (1950), 505-514.

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