Some remarks on the $\mathfrak{S}$-topologies

Peel Man Kim

Introduction

Let F and G be vector spaces over real or complex field, At first, we concider dual system F and G defined with respect to bilinear form$(x, y) \to \langle x, y \rangle$. We can define the weak topology $\sigma(F, G)$ on F, the polar of a subset A of F, and family $\mathfrak{S}$ of $\sigma(F, G)$ bounded subsets of F then we define topology on G.

The purpose of this note is to prove proposition I. II concerning saturated semi-norms and saturated hull of $\mathfrak{S}$

Definition 1

Let F and G be the vector spaces over the real or complex field and $(x, y) \to \langle x, y \rangle$ a bilinear form defined on $F \times G$. We say that F and G form a dual system with respect to the bilinear form if the following conditions are satisfied

i) If for $y \in G$ we have $\langle x, y \rangle = o$ for every $x \in F$, then $y = o$

ii) if for $x \in F$ we have $\langle x, y \rangle = o$ for every $y \in G$, then $x = o$

Definition 2

Given a dual system F, G with respect to the bilinear form $(x, y) \to \langle x, y \rangle$ we can define a locally convex topology $\sigma(F, G)$ on F and a locally convex topology $\sigma(G, F)$ on G. these are called the weak topologies defined by the dual systen F, G.

Remark

A fundamental system of neighborhoods of O in F for the topology $\sigma(F, G)$ is determined by the family of sets such that

$$U_{y_1, y_2, y_n, \varepsilon} = \{x \mid |\langle x, y_k \rangle| \leqq \varepsilon\}$$

where $\varepsilon > 0$ and $(y_k)_{1 \leqq k \leqq n}$ is a finite family of elements of G.

Definition 3

Suppose that the vector spaces F and G form a dual system with respect to the bilinear form $(x, y) \to \langle x, y \rangle$, A be a subset of F. Then the polar of A will be the subset A° of G formed by those elements $y \in G$ which satisfy $\mathrm{Re}\langle x, y \rangle \leqq 1$ for all $x \in A$,

Theorem 1

If F, G is a dual system and A a balanced subset of F, then $A^{\circ\circ}$ is the balanced, convex, $\sigma(F, G)$ closed hull of A.

Proof

Let us remark that if A is a balanced subset of F, then A is a balanced, convex set in G, closed for $\sigma(G, F)$ Indeed, if $|\langle x, y \rangle| \leqq 1$ for $|\lambda| \leqq 1$, than $|\langle x, y \rangle| = |\lambda| \cdot |\langle x, y \rangle| \leqq 1$, hence $A^\circ$is balanced. If $|\langle x, y \rangle| \leqq 1$, $|\langle x, z \rangle| \leqq 1$, $\alpha \geqq o, \beta \geqq o$ $\alpha + \beta = 1$, then $|\langle x, \alpha y + \beta z \rangle| \leqq \alpha |\langle x, y \rangle| + \beta |\langle x, z \rangle| \leqq \alpha + \beta = 1$, i. e, A° is convex. For each $x \in F$ the map $y \to \langle x, y \rangle$ is continuous on G for $\sigma(G, F)$, hence the set $A_x = \{y \mid |\langle x, y \rangle| \leqq 1\}$ is closed as the inverse image of the closed set $|\xi| \leqq 1$ in K. Thus $A^\circ = \bigcap_{x \in A} A_x$ is also closed for $\sigma(F, F)$ Which shows us that $A^{\circ\circ}$is a balanced, convex and $\sigma(F, G)$-closed subset of F.

Let us prove that $A^{\circ\circ}$ is the smallest balanced, convex set containing A which is closed for the topology $\sigma(F, G)$. Thus it is enough to show that if B is a balanced, convex, $\sigma(F, G)$ closed set containing A then $B \supset A^{\circ\circ}$.

Indeed, let $a \notin B$, there exists a continuous linear form f on the real space F_0 underlying the vector space F such that $f(x)<\alpha$ for $x\in B_o$ and $f(a)>\alpha$ Since $0\in B$, we can choose $\alpha=1$. Should F happen to be a complex vector space, we can see that $x\to f(x)$ if (ix) is a linear form on F continuous for $\sigma(F, G)$ In any case there exists a $y\in G$ such that $f(x)=$ Re$\langle x, y\rangle$ and we have Re$\langle x, y\rangle<1$ for all $x\in B$ and Re$\langle a, y\rangle>1$. Since $A\subset B$, it follows from the first inequality that $y\in A^\circ$ and therefore the second inequality implies $a\notin A^{\circ\circ}$

Definition 4

Suppose that the vector spaces F and G form a dual system with respect to the bilinear form$(x, y)\to\langle x, y\rangle$. Let B be a subset of F. Then the absolute polar of B will be the subset B^a of G formed by these elements $y\in G$ which satisfy $|\langle x, y\rangle|\leq 1$ for all $x\in B$

Theorem 2

The absolute polar B^a of a subset B of F is same as the polar A° of the balanced hull A of B.

Theorem 3

Let B be an arbitrary subset of F, then:

a) B^a is a balanced, convex set in G, closed for $\sigma(G, F)$

b) $(\lambda B)^a=\frac{1}{\lambda}B^a$ for $\lambda\in k$, $\lambda\neq 0$, In particular B^a is absorbing iff B is bounded for $\sigma(F, G)$.

Proof

a) Let A be the balanced hull of B. then $B=A^\circ$. By the remark of proof of theorem 1 and theorem 2, B^a is a balanced, convex set in G, closed for $\sigma(G, F)$.

b) The balanced hull of λB is clearly λA, henee $(\lambda B)^a=(\lambda A)^\circ=\frac{1}{\lambda}A^\circ=\frac{1}{\lambda}B^a$.

Furthermore B is bounded iff A is bounded. i. e., iff $A^\circ=B^a$ is absorbing.

Let $\mathfrak{S}$ be a collection of $\sigma(F, G)$-bounded subsets of F. Then the absolute polars B^a of the sets $B\in\mathfrak{S}$ form a collection of absorbing, balanced, convex sets in G.

Thus $\mathfrak{S}$ defines a unique locally convex topology on G. We call it the $\mathfrak{S}$-topology on G. We can define $\mathfrak{S}$-topology on G by the family of semi-norms q_B where $B\in\mathfrak{S}$ and which are given by

$$q_B(y)=\sup_{x\in B}|\langle x, y\rangle|.$$

Definition 5

Let $q_i(1\le i\le n)$ be a finite family of semi-norms on a vector space E. Then its upper envelop q defined by

$$q(x)=\max_{1\leq i\leq n}q_i(x)$$

is also semi-norm. on E and we have

$$\{x|q(x)\le\varepsilon\}=\bigcap_{i=1}^{n}\{x|q_i(x)\le\varepsilon\}=\{x|q_i(x)\leq\varepsilon, 1\le i\le n\}$$

Definition 6

We say that a family of semi-norms on a vector space is saturated if the upper envelop of any finite subfamily also belongs to the family.

Proposition 1

Suppose that the vector spaces F and G form a dual system and let $\mathfrak{S}$ be a collection of balanced, convex, $\sigma(F, G)$-closed, $\sigma(F, G)$-bounded subsets of F. Suppose furthermore that given a finite family of sets in $\mathfrak{S}$, the balanced, convex, $\sigma(F, G)$ closed hull of their union belongs to $\mathfrak{S}$. Then the family of semi-norms $(q_A)_{A\in\mathfrak{S}}$ is saturated.

Proof

By assumption $A=(\bigcup_{1\leq i\leq n}A_i)^{\circ\circ}$ belongs to $\mathfrak{S}$.

Since $(\bigcup_{1\leq i\leq n}A_i)^{\circ\circ}$ is the balanoed, convex, σ (F, G)-closed hull of $\bigcup_{1\leq i\leq n}A_i$,

$$\bigcup_{1\leq i\leq n}A_i\subseteq(\bigcup_{1\leq i\leq n}A_i)^{\circ\circ}=A$$

we have

$$q_A(y)=\sup_{x\in A}|\langle x, y\rangle|=\max_{1\leq i\leq n}q_{A_i}(y).$$

Therefore $q_A(y)$ is the upper envelop of the family of $q_{A_i}(y)$ belongs to the semi-norms defined on $A_i(i=1, 2, \cdots\cdots n)$. The family of semi-norms$(q_A)_{A\in\mathfrak{S}}$ is saturated.

Definition 7

Let F and G be a dual system and $\mathfrak{S}$ a collection of balanced subsets of F. We say that $\mathfrak{S}$ is saturated if the following conditions are satisfied

i) Every subset of a set $A \in \mathfrak{S}$ belongs to $\mathfrak{S}$;

ii) The union of a finite number of sets in $\mathfrak{S}$ belongs to $\mathfrak{S}$

iii) If $A \in \mathfrak{S}$ then $\lambda A \in \mathfrak{S}$ for all $\lambda \neq 0$

iv) The balanced, convex, $\sigma(F. G)$-closed hull of every set in $\mathfrak{S}$ belongs to $\mathfrak{S}$

Definition 8

If $\tilde{\mathfrak{S}}$ is a smallest saturated collection of $\sigma(F. G)$-closed subset containing $\mathfrak{S}$, then $\tilde{\mathfrak{S}}$ is called the saturated hull of $\mathfrak{S}$.

Proposition 2

Let $\mathfrak{S}$ be a collection of $\sigma(F, G)$-bounded subsets of F. Then there exists a saturated hull $\tilde{\mathfrak{S}}$ of $\mathfrak{S}$, Moreover $\tilde{\mathfrak{S}}$-topology on G coincides with the $\mathfrak{S}$-topology.

Proof

Let $\tilde{\mathfrak{S}}$ be thefamily of finite union of bolanced convex $\sigma(F, G)$-closed hull of the sets belonging to $\mathfrak{S}$.

i) If $B \subset (\bigcup_\nu A_\nu)^{\circ\circ}, B^\circ \supset (\bigcap_\nu A_\nu^\circ)^\circ)^\circ$

$$=((\bigcup_\nu A_\nu)^{\circ\circ})^\circ = (\bigcup_\nu A_\nu)^{\circ\circ\circ} = (\bigcup_\nu A_\nu)^\circ$$

Therefore $B \subset B^{\circ\circ} \subset (\bigcup_\nu A_\nu)^{\circ\circ}$.

ii) It is clear that union of a finite number of sets in $\tilde{\mathfrak{S}}$ belcngs to $\tilde{\mathfrak{S}}$.

iii) If $(\bigcup_\nu A_\nu)^{\circ\circ} \in \tilde{\mathfrak{S}}$ then

$$\lambda(\bigcup_\nu A_\nu)^{\circ\circ} = \lambda((\bigcup_\nu A_\nu)^\circ)^\circ$$
$$=(\frac{1}{\lambda}(\bigcup_\nu A_\nu)^\circ)^\circ$$
$$=((\lambda\bigcup_\nu A_\nu)^\circ)^\circ = (\bigcup_\nu \lambda A_\nu)^{\circ\circ}$$

since $\bigcup_\nu \lambda A_\nu$ belongs to $\mathfrak{S}$, $\lambda(\bigcup_\nu A_\nu)^{\circ\circ}$ belongs to $\tilde{\mathfrak{S}}$.

iv) The balancedconvex $\sigma(F, G)$-closed hull of every set $(\bigcup_\nu A_\nu)^{\circ\circ}$ belongs to $\tilde{\mathfrak{S}}$.

It is obvicous that $\tilde{\mathfrak{S}}$ is the smallest saturated· collection of $\sigma(F, G)$-closed subset containing $\mathfrak{S}$.

Bibliography

1. J. Horvath, Topological Vector Spaces, (Lecture Note) 1964.
2. J. L. Kelley, I, Namioka, Linear Topological Spaces, 1963.
3. 엄장일, 局所凸空間論의 諸問題：釜大文理大學報