A THEOREM IN BILATERAL CALCULUS

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§ 1.

The object of the present note is to establish a theorem and to obtain certain bilateral operational relations which are believed to be new.

§ 2. Theorem

THEOREM 1. *Let*

(i) $\phi(p_1, p_2) \doteqdot e^{-2(s_1+s_2)} f(e^{-s_1}, e^{-s_2})$, $\qquad 1/p_r \doteqdot S_r \ (r=1,2)$,

where $L_\pi^2\{f\}$ *is absolutely cconvergent for* $\alpha_i < R(p_i) < \beta_i$,

(ii) $p_1 p_2 f(p_1, p_2) \doteqdot h(x_1, x_2)\, U(x_1, x_2)$,

where $L_\pi^2\{h, U\}$ *is absoluitely convergent for* $R(p_i) > 0$, *then*

$$\frac{4p_1p_2\, q_1q_2\, \phi[(p_1+q_1)/2,\ (p_2+q_2)/2]}{(p_1^2-q_1^2)(p_2^2-q_2^2)\Gamma[(p_1+q_1)/2+2]\,\Gamma[(p_2+q_2)/2+2]}$$

$$\doteqdot e^{-(x_1+y_1)-(x_2+y_2)} h(e^{x_1+y_1},\ e^{x_2+y_2})\, U(x_1+x_2-y_1-y_2),$$

$$0 < 2\alpha_i < R(p_i+q_i) < 2\beta_i;\ R(p_i) > R(q_i);$$

provided that $h(x_1, x_2)$ *is absolutely integrable in* x_i *in* $(0, \infty)$ *and is of the form* $h(x_1, x_2) = \phi(x_i^{1+p_i+\delta},\ x_i^{1+(p_i+q_i)/2+\delta})$ *for small* $x_i, \delta > 0$ *or*

$$\frac{h(x_1,\ x_2)}{x_1^{p_1+2}\ x_2^{p_2+2}} \quad \text{and} \quad \frac{h(x_1,\ x_2)}{x_1^{(p_1+q_1)/2+2}\ x_2^{(p_2+q_2)/2+2}}$$

are absolulely integrable in $(0, \infty)$.

PROOF. We have

$$\frac{\phi(p_1,\ p_2)}{p_1p_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-p_1s_1-p_2s_2} e^{-2s_1-2s_2} f(e^{-s_1},\ e^{-s_2})\, ds_1 ds_2.$$

Putting $e^{-s_i}=t_i$, $i=1,2$, we get

$$\frac{\phi(p_1,\ p_2)}{p_1p_2}=\int_0^\infty \int_0^\infty t_1^{p_1+1}t_2^{p_2+1}f(t_1,\ t_2)dt_1dt_2. \tag{2.3}$$

In (2.3) substituting the value of $t_1t_2f(t_1,\ t_2)$ from (ii), we get

$$\frac{\phi(p_1,\ p_2)}{p_1p_2}=\int_0^\infty \int_0^\infty t_1^{p_1+1}t_2^{p_2+1}dt_1dt_2\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x_1t_1-x_2t_2}h(x_1,\ x_2)\ U(x_1,\ x_2)\ dx_1dx_2$$

$$=\int_0^\infty \int_0^\infty h(x_1,\ x_2)\ dx_1dx_2\int_0^\infty \int_0^\infty t_1^{p_1+1}\ t_2^{p_2+1}\ e^{-x_1t_1-x_2t_2}\ dt_1\ dt_2.$$

$$\frac{\phi(p_1,\ p_2)}{p_1\ p_2}=\Gamma\ (p_1+2)\Gamma\ (p_2+2)\int_0^\infty \int_0^\infty \frac{h(x_1,\ x_2)}{x_1^{p_1+2}x_2^{p_2+2}}\ dx_1dx_2. \tag{2.4}$$

Consider the image integral,

$$I_{p_i,q_i}=p_1p_2q_1q_2\int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-p_1x_1-q_1y_1-p_2x_2-q_2y_2}e^{-(x_1+y_1)-(x_2+y_2)}$$

$$\times h(e^{x_1+y_1},\ e^{x_2+y_2})U(x_1+x_2,\ -y_1-y_2)dx_1dx_2dy_1dy_2$$

$$=p_1p_2q_1q_2\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(p_1-q_1)x_1-(p_2-q_2)x_2}dx_1dx_2$$

$$\times \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x_1+y_1)-(x_2+y_2)}e^{-q_1(x_1+y_1)-q_2(x_2+y_2)}$$

$$\times\ h(e^{x_1+y_1},\ e^{x_2+y_2})\ U(x_1+x_2-y_1-y_2)\ dx_1dx_2dy_1dy_2.$$

Supposing it to exist as an absolutely convergent integral and on making the subtitution $x_i=x_i$ and $x_i+y_i=t_i$, $i=1,2$, We obtain

$$I_{p_i,q_i}=p_1p_2q_1q_2\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-t_1-q_1t_1-t_2-q_2t_2}\ h(e^{t_1},\ e^{t_2})\ dt_1dt_2$$

$$\times\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(p_1-q_1)x_1-(p_2-q_2)x_2}\ U\,[2(x_1+x_2)-t_1-t_2]\ dx_1dx_2$$

$$=p_1p_2q_1q_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(q_1+1)t_1-(q_2+1)t_2}\, h(e^{t_1},\ e^{t_2})\ dt_1dt_2$$

$$\times\int_{\frac{t_1}{2}}^{\infty}\int_{\frac{t_2}{2}}^{\infty} e^{-(p_1-q_1)x_1-(p_2-q_2)x_2}\ dx_1dx_2$$

On putting $e^{t_i}=x_i$, we get

$$I_{p_i,\ q_i}=\frac{p_1p_2q_1q_2}{(p_1-q_1)(p_2-q_2)}\int_0^{\infty}\int_0^{\infty}\frac{h(x_1,\ x_2)\ dx_1dx_2}{x_1^{(p_1+q_1)/2+2}\ x_2^{(p_2+q_2)/2+2}} \tag{2.5}$$

Now on making use of (2.4), we get

$$I_{p_i,q_i}=\frac{p_1p_2q_1q_2\ \ \phi\,[(p_1+q_1)/2,\ [(p_2+q_2)/2]}{(p_1-q_1)(p_2-q_2)\Gamma\,[(p_1+q_1)/2]\,\Gamma\,[(p_2+q_2)/2]\,\Gamma\,[(p_1+q_1)/2+2]\,\Gamma\,[(p_2+q_2)/2+2]}$$

$$\doteqdot e^{-(x_1+y_1)-(x_2+y_2)}\ \ h(e^{x_1+y_1},\ \ e^{x_2+y_2})U(x_1+x_2-y_1-y_2);\quad p_i>q_i$$

§ 3. Applications

(a) Let $e^{-2x}f(e^{-x})=e^{-2x-exp(-x/2)}\doteqdot 2p\Gamma(2p+4)\equiv\phi(p)$,

$$pf(p)=pe^{-\sqrt{p}}\doteqdot\frac{e^{-1/4x}}{2\sqrt{\pi}\ \ x^{3/2}}U(x)\equiv h(x)U(x).$$

Hence from the theorem, we get

$$\frac{2pq\Gamma(p+q+4)}{(p-q)\Gamma\,[(p+q)/2+2]}\doteqdot\frac{1}{2\sqrt{\pi}}e^{-5/2(x+y)-\frac{1}{4}e^{-(x+y)}}\ \ U(x-y),$$

$$R(p)>R(q).$$

(b) Let $e^{-2x}f(e^{-x})=(1+e^{-2x})^{-v/2}e^{-2x}K_v\left(2\sqrt{1+e^{-2x}}\right)\doteqdot(p/2)\Gamma(p/2+1)K^{(2)}_{v-p/2-1}$

$$pf(p)=p(1+p^2)^{-v/2}\left[K_v\left(2\sqrt{1+p^2}\ \right)\right]\doteqdot\sqrt{\frac{\pi}{2}}\,2^{-v}(x^2-4)^{v/2-1/4}J_{v-1/2}\left(\sqrt{x^2-4}\ \right)U(x)$$

$$=h(x)U(x),\ \ x>2.$$

Hence, from the theorem, we get

$$\frac{pq\Gamma[(p+q)/4+1]\ K^{(2)}_{v-(p+q)/4-1}}{(p-q)\Gamma[(p+q)/2+2]} \doteqdot \sqrt{\pi}\, 2^{1/2-v} e^{-(x+y)}\left[e^{2(x+y)}-4\right]^{v/2-1/4}$$

$$\times J_{v-1/2}\left[\left\{e^{2(x+y)}-4\right\}^{1/2}\right]U(x-y),\ R(p)>R(q),\ x+y>\log 2.$$

(c) Let $e^{-2x}f(e^{-x})=\dfrac{e^{-2x}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2F_1(\alpha,\ \beta;\gamma;1/2-e^{-x})$

$$\doteqdot \frac{p\Gamma(p+3)\Gamma(\alpha-p-2)}{(p+2)\Gamma(\gamma-p-2)}\,{}_2F_1(\alpha-p-2,\beta-p-2;\gamma-p-2;1/2)\equiv\phi(p),$$

$$pf(p)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}p\ {}_2F_1\left(\alpha,\beta;\gamma;1/2-p\right)$$

$$\doteqdot x^{(\alpha+\beta-3)/2}\ W_{(\alpha+\beta+1)/2-\gamma,\ (\alpha-\beta)/2}(x)U(x)\equiv h(x)U(x).$$

Hence from the theorem, we get

$$\frac{pq[(p+q)/2+2]\,\Gamma\left(\alpha-\frac{p+q}{2}-2\right)\Gamma\left(\beta-\frac{p+q}{2}-2\right)}{(p-q)\Gamma\left(r-\frac{p+q}{2}-2\right)\Gamma\left(\frac{p+q}{2}+2\right)}$$

$$\times{}_2F_1\left(\alpha-\frac{p+q}{2}-2,\ \beta-\frac{p+q}{2}-2;\ \gamma-\frac{p+q}{2}-2;\frac{1}{2}\right)$$

$$\doteqdot e^{(x+y)(\alpha+\beta-5)/2}\ W_{(\alpha+\beta+1)/2-\gamma,\ (\alpha-\beta)/2}(e^{x+y})U(x-y),\ R(p)>R(q),$$

$$R\left(\begin{matrix}\alpha\\ \beta\end{matrix}-\frac{p+q}{2}\right)>2.$$

(d) Let $e^{-2x}f(e^{-x})=\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(r)}e^{-2x}{}_2F_1(\alpha,\ \beta;\ \gamma;\ 1-e^{-x})$

$$\doteqdot \frac{p\Gamma(p+2)\Gamma(\alpha-p-2)\Gamma(\beta-p-2)\Gamma(p+2+r-\alpha-\beta)}{\Gamma(r-\alpha)\Gamma(\gamma-\beta)}\equiv\phi(p)$$

$\text{Max}[\{0,\ R(\alpha+\beta-\gamma)]<R(p+2)<\min[R(\alpha),R(\beta)].$

$$pf(p)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}p\,{}_2F_1(\alpha,\ \beta;\ \gamma;\ 1-p)$$

$$\doteqdot x^{(\alpha+\beta-3)/2}e^{x/2}W_{\frac{\alpha+\beta+1}{2}-r,\ \frac{\alpha-\beta}{2}}(x)U(x)=h(x)U(x).$$

Hence from the theorem, we get

$$\frac{pq\Gamma\left(\alpha-\frac{p+q}{2}-2\right)\Gamma\left(\beta-\frac{p+q}{2}-2\right)\Gamma\left(\frac{p+q}{2}+r-\alpha-\beta+2\right)}{(p-q)\Gamma(r-\alpha)\Gamma(r-\beta)}$$

$$\doteqdot e^{(x+y)\left(\frac{\alpha+\beta}{2}-5/2\right)} \cdot e^{\frac{1}{2}e^{x+y}} W_{\frac{\alpha+\beta+1}{2}-r,\ \frac{\alpha-\beta}{2}}\left(e^{x+y}\right)U(x-y),$$

$$R(p)>R(q),\ R\left(\frac{p+q}{2}+r-\alpha-\beta\right)>-2,\ R\left(\begin{matrix}\alpha\\\beta\end{matrix}-\frac{(p+q)}{2}\right)>2.$$

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