NOTE ON INFINITESIMAL CL-TRANSFORMATIONS OF NORMAL AND K-CONTACT METRIC SPACES.

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Y. Tashiro and S. Tachibana showed some characteristic properties of Fubinian and C-Fubinian manifolds in their paper [1], where the notion of C-loxodromes was introduced in an almost contact manifold with affine connection. Recently S. Kotō and M. Nagao have obtained invariant tensors under a CL-transformation [2].

Further K. Takamatsu and H. Mizusawa have shown some relations in a compact normal contact metric space under an infinitesimal CL-transformation [3].

In this note, we shall show that an infinitesimal CL-transformation in a normal contact and K-contact metric space has some analogous properties of [3]. In §1, some preliminary notions and identities are given for later use. In §2, we shall deal with a C-loxodrome and a CL-transformation. In §3, infinitesimal CL-transformations in a normal contact metric space will be concerned. In §4, in a K-contact Einstein metric space, an infinitesimal CL-transformation necessary projective.

§ 1. Preliminaries

An $n(=2m+1)$-dimensional differentiable manifold M of class C^∞ with (φ, ξ, η, g)-structure (or an almost contact metric structure) has been defined by S. Sasaki [4]. By definition it is a manifold with tensor fields $\varphi_j{}^i, \xi^i, \eta_i$ and so called an associated Riemannian metric tensor g_{ji} defined over M which satisfy the following relations:

(1. 1) $\qquad \xi^i\eta_i=1,$

(1. 2) $\qquad \mathrm{rank}\ |\varphi_j{}^i|=n-1,$

(1. 3) $\qquad \varphi_j{}^i\xi^j=0,$

(1. 4) $\qquad \varphi_j{}^i\eta_i=0,$

(1. 5) $\qquad \varphi_j{}^r\varphi_r{}^i=-\delta_j^i+\xi^i\eta_j,$

(1. 6) $g_{ji}\ \xi^j = \eta_i,$

(1. 7) $g_{ji}\varphi_k{}^j\varphi_h{}^i = g_{kh} - \eta_k\eta_h.$

On the other hand let M be a differentiable manifold with a contact structure. If we put

(1. 8) $2g_{ir}\varphi_j{}^r = 2\varphi_{ji} = \partial_j\eta_i - \partial_i\eta_j,$

then we can find four tensors $\varphi_j{}^i, \xi^i, \eta_i$ and g_{ji} so that they define an (φ, ξ, η, g)-structure. Such a structure is called a contact metric structure [4].

In an almost contact metric space there are four tensor fields $N_{ji}{}^h, N_j{}^i, N_{ji}$ and N_j which are the analogue of the Nijenhuis tensor in an almost complex structure [4]. In a contact metric space, $N_j{}^i = 0$ and $N_{ji} = 0$ hold good, $N_j{}^i = 0$ is equivalent to the fact ξ^i is a Killing vector field and $N_{ji}{}^h = 0$ yields $N_j{}^i = 0$.

A contact metric space with $N_{ji} = 0$ or $N_{ji}{}^h = 0$ is called a K-contact metric space or a normal contact metric space respectively. Of course a normal contact metric space is a K-contact metric space and a K-contact metric space is a contact metric space [6]. In the following we consider a notation η^i instead of ξ^i.

A K-contact metric space in which the Ricci tensor takes the form

(1. 9) $R_{ji} = ag_{ji} + b\eta_j\eta_i;$

is called a K-contact η-Einstein space, where a and b become constant $(n>3)$, and

(1.10) $a + b = n - 1, \quad R = an + b$

hold good [5], [6].

Let $R_{kji}{}^h$ be the Riemannian curvature tensor and put

(1.11) $H_{ji} = \varphi^{kh}R_{kjih},$ then $H_{ji} = -\frac{1}{2}\varphi^{kh}R_{khji}.$

In a contact metric space, φ_{ji} is a skew symmetric closed tensor and

(1.12) $\nabla_r\varphi_j{}^r = (n-1)\eta_j$

holds good, where ∇_i denotes the covariant differentiation with respect to the Riemannian connection.

In a K-contact metric space the following identities are valid [6].

(1.13) $\nabla_j\eta_i = \varphi_{ji},$

(1.14) $\nabla_k \varphi_{ji} + R_{rkji}\eta^r = 0,$

(1.15) $R_{kjih}\eta^k \eta^j = 0,$

(1.16) $R_{kjih}\eta^k \eta^h = g_{ji} - \eta_j \eta_i,$

(1.17) $R_{ir}\eta^r = (n-1)\eta_i.$

In a normal contact metric space

(1.18) $\nabla_k \varphi_{ji} = \eta_j g_{ki} - \eta_i g_{kj},$

(1.19) $\eta_r R_{kji}{}^r = \eta_k g_{ji} - \eta_j g_{ki},$

(1.20) $\varphi_j{}^r R_{ri} = H_{ji} + (n-2)\varphi_{ji},$

and also (1.13), (1.17) hold good [6].

§ 2. C-loxodromes and infinitesimal CL-transformations ([1], [3]).

The equation of a C-loxodrome in a normal contact metric space in terms of any prameter t is

(2. 1)
$$\frac{\delta^2 x^h}{dt^2} = \alpha \frac{dx^h}{dt} + a\eta_j \varphi_i{}^h \frac{dx^j}{dt} \frac{dx^i}{dt},$$

where δ is indicates the covariant differentiation along the curve $x^i(t)$, α is a function of t and a is a constant.

Now let us consider a relation between symmetric affine connections in an almost contact manifold. If it carries C-loxodrames to C-loxodromes, then it will be called a CL-transformation. By the usual process it follows that their connections are in the relation

$$'\Gamma_{ji}^h - \Gamma_{ji}^h = \rho_j \delta_i^h + \rho_i \delta_j^h + \alpha(\eta_j \varphi_i{}^h + \eta_i \varphi_j{}^h),$$

where ρ_i is a vector field and α is a certain scalar [1].

In a normal contact or K-contact metric space a vector v^i is called an infinitesimal CL-transformation if it satisfies

(2. 2)
$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \rho_j \delta_i^h + \rho_i \delta_j^h + \alpha(\eta_j \varphi_i{}^h + \eta_i \varphi_j{}^h),$$

where $\underset{v}{\pounds}$ is the operator of Lie derivative and $\left\{ {h \atop ji} \right\}$ is Riemannian connection. Contracting h and j in (2.2), we see that ρ_i is a gradient.

In a normal contact metric space an infinitesimal CL-transformation hold good the following relations [3].

$$\underset{v}{\pounds} R_{ji}=(1-n)\nabla_j\rho_i+2\alpha(n\eta_j\eta_i-g_{ji})+\eta_j\varphi_i{}^r\nabla_r\alpha+\eta_i\varphi_j{}^r\nabla_r\alpha, \tag{2.3}$$

$$\eta_h\underset{v}{\pounds} R_{kji}{}^h=\eta_j\nabla_k\rho_i-\eta_k\nabla_j\rho_i+\alpha(\eta_k g_{ji}-\eta_j g_{ki}). \tag{2.4}$$

Taking the Lie derivative of the both sides of (1.19) and substituting (2.4) into the equation thus obtained, we get

$$R_{kji}{}^h\underset{v}{\pounds}\eta_h=g_{ji}\underset{v}{\pounds}\eta_k+\eta_k\underset{v}{\pounds}g_{ji}-g_{ki}\underset{v}{\pounds}\eta_j-\eta_j\underset{v}{\pounds}g_{ki}-\eta_j\nabla_k\rho_i$$
$$+\eta_k\nabla_j\rho_i+\alpha(\eta_j g_{ki}-\eta_k g_{ji}).$$

Transvecting the last equation with g^{ji}, we have

$$R_k{}^h\underset{v}{\pounds}\eta_h=(n-1)\underset{v}{\pounds}\eta_k+\eta_k(g^{ji}\underset{v}{\pounds}g_{ji}+\nabla_r\rho^r)-\eta^r(\underset{v}{\pounds}g_{kr}+\nabla_k\rho_r)$$
$$+\alpha\eta_k(1-n). \tag{2.5}$$

Finally we shall prepare the following two theorems which have been proved by H. Mizusawa and K. Takamatsu.

LEMMA 2.1. *In a normal contact metric space, if v^i is an infinitesimal CL-transformation, then the following relation holds good* [3].

$$\underset{v}{\pounds}g_{ji}=-\nabla_j\rho_i+\alpha(g_{ji}+\eta_j\eta_i). \tag{2.6}$$

LEMMA 2.2. *In a normal contact metric space of constant scalar curvature, the relation*

$$\nabla^i H_{ji}=[R-(n-1)^2]\,\eta_j \tag{2.7}$$

holds good [6].

§3. Infinitesimal CL-transformations in a normal contact metric space.

Let v^i be an infinitesimal CL-transformation in a normal contact metric space. Substituting (2.2) and (2.6) into the identity

$$\nabla_k\underset{v}{\pounds}g_{ji}=g_{hi}\underset{v}{\pounds}\left\{{h \atop kj}\right\}+g_{jh}\underset{v}{\pounds}\left\{{h \atop ki}\right\},$$

and using (1.13), we get

$$-\nabla_k\nabla_j\rho_i+(g_{ji}+\eta_j\eta_i)\nabla_k\alpha=2\rho_k\rho_{ji}+\rho_j g_{ki}+\rho_i g_{jk}. \tag{3.1}$$

By virtue of Ricci identity, this equation is written as

$$R_{kjir}\rho^r+(g_{ji}+\eta_j\eta_i)\nabla_k\alpha-(g_{ki}+\eta_k\eta_i)\nabla_j\alpha=\rho_k g_{ji}-\rho_j g_{ki}. \tag{3.2}$$

Transvecting (3.2) with g^{kh}, we have

$$R_{rij}{}^h\rho^r+(g_{ji}+\eta_j\eta_i)\nabla^h\alpha-(\delta_i^h+\eta^h\eta_i)\nabla_j\alpha=\rho^h g_{ji}-\rho_j\delta_i^h. \tag{3.3}$$

Moreover, transvecting (3.1) with g^{ih}, we get

$$-\nabla_k\nabla_j\rho^h+(\delta_j^h+\eta^h\eta_j)\nabla_k\alpha=2\rho_k\delta_j^h+\rho_j\delta_k^h-\rho^h g_{kj}. \tag{3.4}$$

According to (3.3) and (3.4), it follows that

$$\underset{\rho}{\pounds}\left\{{h \atop kj}\right\}+2(\rho_k\delta_j^h+\rho_j\delta_k^h)=(\delta_k^h+\eta^h\eta_k)\nabla_j\alpha+(\delta_j^h+\eta^h\eta_j)\nabla_k\alpha -(g_{jk}+\eta_j\eta_k)\nabla^h\alpha. \tag{3.5}$$

Thus we have the following

PROPOSITION 3.1. *Let v^i be an infinitesimal CL-transformation and ρ_i be its associated vector. If α is constant then ρ^i is an infinitesimal projective transformation.*

Next we shall prove the following proposition.

PROPOSITION 3.2.* *Let M be a normal contact metric space of constant scalar curvature $R\neq n(n-1)$ and ρ_i be an associated vector of an infinitesimal CL-transformation, then $\eta^r\rho_r=0$.*

PROOF. Contracting h and i in (3.3), we have

$$-R_{rj}\rho^r+\eta_j\eta_r\nabla^r\alpha-n\nabla_j\alpha=\rho_j-n\rho_j.$$

Transvecting the last equation with η^j and using of (1.17), we get

$$\eta^r\nabla_r\alpha=0 \text{ for } n>1. \tag{3.6}$$

On the other hand, transvecting (3.2) with φ^{ji} and using (1.4) and (1.11), we have

$$H_{kr}\rho^r=\varphi_k{}^r(\rho_r-\nabla_r\alpha). \tag{3.7}$$

* This result is also obtained by K. Takamatsu and H. Mizusawa in compact case.

Operating ∇^k to (3.7) and making use of (1.12) and (3.6), we get

$$\rho^r \nabla^i H_{ir} = (1-n)\rho^r \eta_r. \tag{3.8}$$

According to (3.8) and (2.7), it follows that

$$[R-n(n-1)]\eta^r \rho_r = 0.$$

This completes the proof by Lemma 2.2.

Q. E. D.

K. Takamatsu and H. Mizusawa have proved that, in an $n(n>3)$-dimensional compact normal contact metric space, an infinitesimal CL-transformation is necessary projective.

Now we shall prove the following theorem.

THEOREM 3.3. *In an $n(n>1)$-dimensional normal contact metric space an infinitesimal CL-transformation with $\eta_r \rho^r = 0$ is necessary projective.*

PROOF. Substituting (2.6) into (2.5), we get

$$R_k{}^h \underset{v}{\pounds} \eta_h = (n-1) \underset{v}{\pounds} \eta_k. \tag{3.9}$$

On the other hand (1.17) yields that

$$R_k{}^h \underset{v}{\pounds} \eta_h + \eta_h \underset{v}{\pounds} R_k{}^h = (n-1) \underset{v}{\pounds} \eta_k.$$

Thus we have

$$\eta_h \underset{v}{\pounds} R_k{}^h = \eta_h R_{kr} \underset{v}{\pounds} g^{hr} + \eta^r \underset{v}{\pounds} R_{kr} = 0. \tag{3.10}$$

From (2.6), we obtain

$$\underset{v}{\pounds} g^{hr} = \nabla^r \rho^h - \alpha(g^{hr} + \eta^h \eta^r),$$

and substituting this and (2.3) into (3.10), we get

$$R_k{}^r \eta_h \nabla_r \rho^h + (1-n)\eta_h \nabla_k \rho^h + \varphi_k{}^r \nabla_r \alpha = 0. \tag{3.11}$$

By hypothesis, i.e., $\eta_h \rho^h = 0$, we get

$$\eta_h \nabla_k \rho^h = -\rho^h \nabla_k \eta_h = \rho^h \varphi_{hk}. \tag{3.12}$$

Substituting (3.12) into (3.11), we have

$$\rho^h \varphi_{hr} R_k{}^r + (1-n)\rho^h \varphi_{hk} + \varphi_k{}^r \nabla_r \alpha = 0.$$

By (1.20), this equation can be written as

$$H_{hk}\rho^h - \varphi_{hk}\rho^h + \varphi_k{}^r\nabla_r\alpha = 0.$$

From (3.7) and the last equation, it follows that

$$(3.13) \qquad \varphi_k{}^r\nabla_r\alpha = 0.$$

Transvecting (3.13) with $\varphi_j{}^k$ and using (1.5) and (3.6), we get

$$\nabla_k\alpha = 0, \quad \text{i.e.} \quad \alpha = \text{constant}.$$

This completes the proof by proposition 3.1.

Q. E. D.

According to proposition 3.2 and theorem 3.3, we have the following.

COROLLARY. *In an $n(n>1)$-dimensional normal contact metric space with constant scalar curvature $R\neq n(n-1)$, an infinitesimal CL-transformation is necessary projective.*

§ 4. Infinitesimal CL-transformations in a K-contact metric space.

Let v^i be an infinitesimal CL-transformation in a K-contact metric space.

LEMMA 4.1. *In a K-contact metric space if v^i is an infinitesimal CL-transformation, then* (2.3) *and* (2.4) *hold good.*

PROOF. From (2.2) we have

$$(4.1) \qquad \nabla_k \underset{v}{\pounds}\left\{{h \atop ji}\right\} = \delta_i^h\nabla_k\rho_j + \delta_j^h\nabla_k\rho_i + \alpha(\varphi_i{}^h\nabla_k\eta_j + \eta_j\nabla_k\varphi_i{}^h + \varphi_j{}^h\nabla_k\eta_i + \eta_i\nabla_k\varphi_j{}^h) + (\eta_j\varphi_i{}^h + \eta_i\varphi_j{}^h)\nabla_k\alpha.$$

Substituting (4.1) into the following identity

$$\underset{v}{\pounds}R_{kij}{}^h = \nabla_k\underset{v}{\pounds}\left\{{h \atop ji}\right\} - \nabla_j\underset{v}{\pounds}\left\{{h \atop ki}\right\},$$

and using of (1.13) and (1.14), we get

$$(4.2) \qquad \underset{v}{\pounds}R_{kji}{}^h = \delta_j^h\nabla_k\rho_i - \delta_k^h\nabla_j\rho_i + \alpha\{2\varphi_{kj}\varphi_i{}^h + \varphi_{ki}\varphi_j{}^h - \varphi_{ji}\varphi_k{}^h - \eta_j\eta^r R_{rki}{}^h + \eta_k\eta^r R_{rji}{}^h - \eta_i\eta^r R_{rkj}{}^h + \eta_i\eta^r R_{rjk}{}^h\} + \varphi_i{}^h(\eta_j\nabla_k\alpha - \eta_k\nabla_j\alpha) + \eta_i(\varphi_j{}^h\nabla_k\alpha - \varphi_k{}^h\nabla_j\alpha).$$

Contracting h and k in (4.2) and using of (1.16) and (1.17), we have

$$\underset{v}{\pounds}R_{ji}=(1-n)\nabla_j\rho_i+\alpha\{2(-g_{ji}+\eta_j\eta_i)+2(n-1)\eta_j\eta_i\}$$
$$+\eta_j\varphi_i{}^r\nabla_r\alpha+\eta_i\varphi_j{}^r\nabla_r\alpha.$$

Hence, we have (2.3).

Similarly transvecting (4.2) with η_h, we get (2.4).

Q. E. D.

Transvecting (2.4) with η^k, it can be written as

$$\eta^k\eta_h\underset{v}{\pounds}R_{kji}{}^h=\eta_j\eta_r\nabla^r\rho_i-\nabla_j\rho_i+\alpha(g_{ji}-\eta_j\eta_i). \tag{4.3}$$

Taking the Lie derivative of the both sides of (1.16) and substituting (4.3) into the equation obtained, we get

$$\underset{v}{\pounds}g_{ji}=\eta_j\eta^r\nabla_r\rho_i-\nabla_j\rho_i+\alpha(g_{ji}-\eta_j\eta_i)+\eta^kR_{kji}{}^h\underset{v}{\pounds}\eta_h$$
$$+\eta_hR_{kji}{}^h\underset{v}{\pounds}\eta^k+\eta_j\underset{v}{\pounds}\eta_i+\eta_i\underset{v}{\pounds}\eta_j. \tag{4.4}$$

Next, from (1.9) we have

$$\underset{v}{\pounds}R_{ji}=a\underset{v}{\pounds}g_{ji}+b(\eta_i\underset{v}{\pounds}\eta_j+\eta_j\underset{v}{\pounds}\eta_i). \tag{4.5}$$

Substituting (2.3), (1.10) and (4.4) into (4.5), we get

$$(1-n)\nabla_j\rho_i+2\alpha(n\eta_j\eta_i-g_{ji})+\eta_j\varphi_i{}^r\nabla_r\alpha+\eta_i\varphi_j{}^r\nabla_r\alpha$$
$$=a\{\eta_j\eta^r\nabla_r\rho_i-\nabla_j\rho_i+\alpha(g_{ji}-\eta_j\eta_i)+\eta^kR_{kji}{}^h\underset{v}{\pounds}\eta_h+\eta_hR_{kji}{}^h\underset{v}{\pounds}\eta^k\} \tag{4.6}$$
$$+(n-1)(\eta_j\underset{v}{\pounds}\eta_i+\eta_i\underset{v}{\pounds}\eta_j).$$

Transvecting (4.6) with g^{ji}, we have

$$(1-n)\nabla^r\rho_r=a\{\beta-\nabla^r\rho_r+\alpha(n-1)\}+2(n-1)\eta^r\underset{v}{\pounds}\eta_r, \tag{4.7}$$

where $\beta=\eta^r\eta^s\nabla_r\rho_s$.

On the other hand, (1.17) yields that

$$\eta^j\underset{v}{\pounds}R_{ji}+R_{ji}\underset{v}{\pounds}\eta^j=(n-1)\underset{v}{\pounds}\eta_i. \tag{4.8}$$

Substituting (2.3) into (4.8) and transvecting with η^i, we have

$$(1-n)\eta^r\eta^s\nabla_r\rho_s+2\alpha(n-1)=2(n-1)\eta^r\underset{v}{\pounds}\eta_r,$$

or

$$2\alpha-\beta=2\eta^r\underset{v}{\pounds}\eta_r, \qquad \text{for } n>1. \tag{4. 9}$$

From (4.7) and (4.9) it follows that

$$\{a-(n-1)\}\nabla^r\rho_r=(n-1)(a+2)\alpha+\{a-(n-1)\}\beta.$$

Suppose $a=n-1$(i.e. Einstein metric space) then the last equation can be written as $(n-1)^2\alpha=0$. Therefore $\alpha=0$ for $n>1$.

Thus we have the following theorem.

THEOREM 4.2. *In a K-contact Einstein metric space, an infinitesimal CL-transformation is necessary projective.*

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