A CHARACTERIZATION OF BAER LOWER RADICAL PROPERTY

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For each ring R, let $D_1(R)$ be the set of all ideals of R, and by induction, we define $D_{n+1}(R)$ to be the family of all rings which are ideals of some ring in $D_n(R)$ and let

$$D(R) = \bigcup \{D_n(R) : n=1, 2, 3, \cdots\cdots\}$$

A ring R is called an $\mathscr{L}$-ring if $D(R/I)$ contains a non-zero nilpotent ring for each ideal I of R and $I \neq R$. We note that each nilpotent ring is an $\mathscr{L}$-ring, and every isomorphic image of an $\mathscr{L}$-ring is an $\mathscr{L}$-ring.

THEOREM. *A ring R is an $\mathscr{L}$-ring if, and only if R is a Baer lower radical ring.*

To prove the theorem, we summerize the facts we need [1]. Let $\mathfrak{S}$ be a class of rings. We shall say that the ring R is an $\mathfrak{S}$-*ring* if R is in $\mathfrak{S}$. An ideal J of R will be called an $\mathfrak{S}$-*ideal* if J is an $\mathfrak{S}$-rng. A ring which does not contain any non-zero $\mathfrak{S}$-ideals will be called $\mathfrak{S}$-*semi-simple.* We shall call $\mathfrak{S}$ *a radical property* if the following three conditions hold:

(A) A homomorphic image of an $\mathfrak{S}$-ring is an $\mathfrak{S}$-ring.

(B) Every ring R contains a largest $\mathfrak{S}$-ideal S.

(C) The quotient ring R/S is $\mathfrak{S}$-semi-simple.

LEMMA 1. *A class $\mathfrak{S}$ of rings is a radical property if and only if*

(A) A homomorphic image of an $\mathfrak{S}$-ring is an $\mathfrak{S}$-ring

(D) If every non-zero homomorphic image of a ring R contains a non-zero $\mathfrak{S}$-ideal, then R is an $\mathfrak{S}$-ring.

LEMMA 2. *The Baer lower radical property $\mathfrak{B}$ is the lower radical property determined by the class of all nilpotent ring, i.e., if $\mathfrak{S}$ is a radical property and every nilpotent ring is an $\mathfrak{S}$-ring then every $\mathfrak{B}$-ring is an $\mathfrak{S}$-ring.*

LEMMA 3. *If R has no non-zero nilpotent ideals and C is an ideal of R then C has no non-zero nilpotent ideals.*

PROOF of THEOREM. First we shall show that $\mathscr{L}$ is a radical property and since every nilpotent ring is $\mathscr{L}$-ring hence by LEMMA 2, every $\mathfrak{B}$-ring is an $\mathscr{L}$-ring.

If R is an $\mathscr{L}$-ring and I is any ideal of R. Consider the quotient ring R/I, and any proper ideal J/I of R/I.

$$R/I/J/I \cong R/J$$

By defintion, $D(R/J)$ contains a non-zero nilpotent ring, therefore $D(R/I/J/I)$ contains a non-zero nilpotent ring and hence R/I is an $\mathscr{L}$-ring. Since every homomorphic image of R is isomorphic with R/I for some I, hence (A) follows.

Suppose that every non-zero homomorphic image of R contains a non-zero $\mathscr{L}$-ideal, and let I be any ideal of R and $I \neq R$. Then R/I contains a non-zero $\mathscr{L}$-ideal J/I. Now $D(J/I)$ contains a non-zero nilpotent ring and clearly $D(J/I) \subset D(R/I)$, hence $D(R/I)$ contains a non-zero nilpotent ring. By definition of $\mathscr{L}$-ring, R is an $\mathscr{L}$-ring. This proves (D). By LEMMA 1, $\mathscr{L}$ is a radical property.

To show $\mathscr{L} \subseteq \mathfrak{B}$, let R be an $\mathscr{L}$-ring and let I be an ideal of R and $I \neq R$. Then $D(R/I)$ contains a non-zero nilpotent ring J/I. By LEMMA 3, if R/I contains no non-zero nilpotent ideals then $D_1(R/I)$ contains no non-zero nilpotent rings and by induction, $D_n(R/I)$ contains no non-zero nilpotent rings and hence $D(R/I)$ contains no non-zero nilpotent rings. This contradiction shows that R/I contains a non-zero nilpotent ideal which is a $\mathfrak{B}$-ideal. By (D), R is a $\mathfrak{B}$-ring.

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BIBLIOGRAPHY

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