TIGHT FUNCTIONALS AND THE STRICT TOPOLOGY

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Let X be any Hausdorff space, $C(X)$ the linear space of all bounded (real or complex valued) functions on X. The supremum norm of a function f on X is written as $\|f\|$.

For locally compact X, R.C. Buck [1] (see also [2], [3]) has introduced the strict topology in $C(X)$, and proved that every strictly continuous linear functional on $C(X)$ can be represented by a Radon measure. Our purpose is to extend this definition to general topological spaces X and to prove that a linear functional on $C(X)$ is strictly continuous if and only if it is tight. Here a linear functional I on $C(X)$ is called *tight* (see [4], [5]) if $\lim I(f_\alpha)=0$ for every norm-bounded net $(f_\alpha)_{\alpha\in A}$ in $C(X)$ that converges to 0 uinformly on all compact subsets of X.

Let Φ be the set of all bounded real functions φ on X that vanish at infinity (i.e., for every $\varepsilon>0$ the set $\{x\in X: |\varphi(x)|>\varepsilon$ is contained in a compact subset of X). Every $\varphi\in\Phi$ induces a seminorm $\| \ \|_\varphi$ on $C(X)$ by the formula

$$\|f\|_\varphi=\|f_\varphi\|, \quad (f\in C(X)).$$

The *strict topology* in $C(X)$ is the locally convex Hausdorff topology determined by the collection of these seminorms. It is easy to see that for locally compact X we obtain the same topology in $C(X)$ if we admit only *continuous* functions in to Φ. Thus, for locally compact X our definition coincides with Buck's.

For any function f on X and any $K\subset X$ we denote by $f|K$ the restriction of f to K, while $\|f|K\|=\sup\{|f(x)|: x\in K\}$.

THEOREM. *A linear functional I on $C(X)$ is tight if and only if it is continuous with respect to the strict topology in $C(X)$.*

PROOF. "IF". Let I be strictly continuous. Let $(f_\alpha)_{\alpha\in A}$ be a not in $C(X)$, $\|f_\alpha\|\le 1$ for every α, $\lim f_\alpha=0$ uniformly on compacts. We have to prove that $\lim I(f_\alpha)=0$. Take any $\varepsilon>0$. There is a $\varphi\in\Phi$ such that $|I(f)|\le\|f\|_\varphi$ for evey $f\in C(X)$ and there is a compact set $K\subset X$ that contains every $x\in X$ for which $|\varphi(x)|>\varepsilon$. For sufficiently large α we have $\|f_\alpha|K\|\cdot\|\varphi\|\le\varepsilon$; for all these α

$$|f_\alpha(x)\varphi(x)|\leq|f_\alpha(x)|\cdot\|\varphi\|\leq\varepsilon \text{ if } x\in K,$$

$$|f_\alpha(x)\varphi(x)|\leq\|f_\alpha\|\varepsilon\leq\varepsilon \text{ if } x\notin K.$$

Hence, $|I(f_\alpha)|\leq\|f_\alpha\|_\varphi\leq\varepsilon$ for sufficiently large α.

"ONLY IF". Now assume that I is tight. We construct a $\varphi\in\Phi$ such that $|I(f)|\leq\|f\|_\varphi$ for every $f\in C(X)$.

We first remark that for any given $\varepsilon>0$ there exists a compact set $K\subset X$ such that $|I(f)|\leq\varepsilon\|f\|$ for all $f\in C(X)$ that vanish on K. If for some ε no such K existed, for every compact K there would be an $f_K\in C(X)$ with $f_K|K=0$, $\|f_K\|\leq 1$, $|I(f_K)|>\varepsilon$. Now the compact subsets K of X form in a natural way a directed set ($K\leq K'$ if $K\subset K'$). Thus, the f_K form a norm-bounded net in $C(X)$ that converges to 0 uniformly on compacts. It follows that $\lim I(f_K)=0$, and we have obtained a contradiction.

Hence, for every positive integer n there is a compact $K_n\subset X$ such that $|I(f)|\leq 4^{-n}\|f\|$ if $f\in C(X)$, $f|K_n=0$. We may assume $K_1\subset K_2\subset\cdots\cdots$. For every $f\in C(X)$ and every n define $\theta_n f\in C(X)$ by

$$(\theta_n f)(x)=f(x) \text{ if } |f(x)|\leq\|f|K_n\|,$$

$$(\theta_n f)(x)=\frac{f(x)}{|f(x)|}\|f|K_n\| \text{ if } |f(x)|\geq\|f|K_n\|.$$

Then $\|\theta_n f\|\leq\|f|K_n\|$, $\theta_n f=f$ on K_n, and $|f|=|\theta_n f|+|f-\theta_n f|$.

Let φ_n be the characteristic function of K_n, and put $\varphi=\sum 2^{-n+2}\varphi_n$. Clearly, $\varphi\in\Phi$, and we are done if $|I(f)|\leq\|f\|_\varphi$ for all $f\in C(X)$.

Take $f=f_0\in C(X)$. For every n define f_n, $g_n\in C(X)$ by

$$g_n=\theta_n f_{n-1},\ f_n=f_{n-1}-g_n.$$

Then $f_n|K_n=0$, and $|f_n|\leq|f_{n-1}|\leq\cdots\cdots\leq|f_0|=|f|$, so that $|I(f_n)|\leq 4^{-n}\|f_n\|\leq 4^{-n}\|f\|$. Because both f_n and f_{n-1} vanish on K_{n-1}, so does g_n, and $|I(g_n)|\leq 4^{-n+1}\|g_n\|=4^{-n+1}\|\theta_n f_{n-1}\|\leq 4^{-n-1}\|f_{n-1}|K_n\|\leq 4^{-n-1}\|f|K_n\|$. Since $f=g_1+\cdots\cdots+g_n+f_n$ for every n, $|I(f)|\leq\sum 4^{-n+1}\|f|K_n\|$.

Now for $x\in K_n$ we have $2^{-n+2}|f(x)|\leq|f(x)\varphi(x)|\leq\|f\|_\varphi$.
Therefore, $\|f|K_n\|\leq 2^{n-2}\|f\|_\varphi$. It follows that $|I(f)|\leq\|f\|_\varphi$.

NOTE. A simiar procedure works for any linear space E of bounded functions on X provided it has the following property. For every compact $K \subset X$ there is a map $\theta: E \to F$ and a number β such that for all $f \in E$: $\theta(f) = f$ on K, $|\theta(f)| \leq \beta |f|$, $\|\theta(f)\| \leq \beta \|f|K\|$.

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