RADICAL PROPERTIES AND PARTITIONS OF RINGS

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§ 1. Introduction.

Let S be a class of rings. We shall say that a ring R is an *S-ring* if R is in S. An ideal J of a ring R is called an *S-ideal* if J is an S-ring. A ring which contains no non-zero S-ideal is said to be *S-semi-simple*. We shall call *S a radical property* if the following three conditions hold:

(A) A homomorphic image of an S-ring is an S-ring.

(B) Every ring R contains a largest S-ideal $S(R)$.

(C) The quotient ring $R/S(R)$ is S-semi-simple.

The largest S-ideal $S(R)$ of a ring R is called the S-radical of R; and if $S(R) = R$, then R is called an S-radical ring [1], [2], [3].

Let S be a radical property, and let $\mathfrak{a}$ be the class of all simple rings. Then S partitions $\mathfrak{a}$ into two disjoint classes: $\mathfrak{a}_1$, the upper class, consisting of all S-semi-simple rings of $\mathfrak{a}$; and $\mathfrak{a}_2$, the lower class consisting of all S-radical rings of $\mathfrak{a}$. Conversely, if $(\mathfrak{a}_1, \mathfrak{a}_2)$ is any partition of $\mathfrak{a}$ with isomorphic rings in the same class, then there exists a radical property S such that every ring in $\mathfrak{a}_1$ is an S-semi-simple ring, and every ring in $\mathfrak{a}_2$ is an S-radical ring [2], [3]. However, the class of all simple rings is not the only class having this property. The purpose of this paper is to extend the class of rings enjoying this property.

§ 2.

Given a class M of rings with the property that every non-zero ideal of a ring in M can be mapped homomorphically onto some non-zero ring of M, let S_M be the class of all rings which cannot be mapped homomorphically onto any non-zero ring of M. Then S_M, the upper radical property determined by M, is the largest radical property such that every ring in M is semi-simple.

In [4] the lower radical property was constructed for any class $\mathfrak{a}$ of rings. The construction was as follows: Let $\mathfrak{a}$ be a class of rings from a category C of rings, and let $\bar{\mathfrak{a}}$ be the class of all homomorphic images of rings in $\mathfrak{a}$. For each ring R in C, let $D_1(R)$ be the set of all ideals of R. Inductively we define

$D_{n+1}(R)$ to be the set of all rings which are ideals of rings in $D_n(R)$, i.e., $Q \in D_{n+1}(R)$ if and only if Q is an ideal of a ring in $D_n(R)$. Setting $D(R)=\bigcup_{n=1}^{\infty} D_n(R)$, a ring R is called an $S_{\mathcal{A}}$-ring if $D(R/I)$ contains a non-zero ring which is isomorphic to a ring in $\bar{\mathcal{A}}$ for each ideal I of R and $I \neq R$. Then $S_{\mathcal{A}}$ is the smallest radical property for which every ring in $\mathcal{A}$ is a radical ring.

THEOREM. *Let $\mathcal{A}$ be a class of rings which satisfies the following two conditions*:

(1) *If $R \in \mathcal{A}$ and I is a proper ideal of R, then I is isomorphic to R.*

(2) *If $R \in \mathcal{A}$ and R/I is not isomorphic to R, then R/I is not isomorphic to any ring in $\mathcal{A}$.*

Then for any radical property S, every non-zero ring in $\mathcal{A}$ is either an S-radical ring or an S-semi-simple ring. Moreover, for any partition $(\mathcal{A}_1, \mathcal{A}_2)$ of $\mathcal{A}$ *with isomorphic rings in the same class; if S is the upper radical property determined by $\mathcal{A}_1$ or the lower radical property determined by $\mathcal{A}_2$, then each ring in $\mathcal{A}_1$ is S-semi-simple and each ring in $\mathcal{A}_2$ is S-radical.*

NOTE. The following are examples of rings which satisfy (1) and (2).

1. The class of all simple rings.
2. Any class $\mathcal{B}$ of simple rings.
3. $\mathcal{B} \cup \{C^{\infty}\}/(\{Z_p : p=2, 3, 5, \cdots\} \cup \{R: R \text{ is isomorphic to } Z_p \text{ for some prime } p\})$, where C^{∞} is the zero ring of integers, and Z_p is the zero ring of integers modulo a prime number p.

We also note:

(3) If a ring R has property (1), and R is isomorphic to a ring R', then R' also has property (1).

In the proof of the theorem we employ the following notation.

$R \approx R'$ denotes: The rings R and R' are isomorphic.

$I \leq R$ denotes: I is an ideal of the ring R.

O, (depending upon the context in which it appears), denotes: either the zero ring or the zero ideal of a ring.

PROOF OF THEOREM. By (1), for any radical property S, every non-zero ring R in $\mathcal{A}$ is either S-radical or S-semi-simple. For let $0 \neq R \in \mathcal{A}$, then since $S(R)$ is an ideal of R, we have by (1) that $S(R) \approx R$, in which case R is S-radical; or $S(R)=0$, in which case R is S-semi-simple.

Next, let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a partition of $\mathfrak{a}$ with isomorphic rings in the same class, and let S_{a_2} be the lower radical property (as constructed in [4] determined by $\mathfrak{a}_2$. Then each ring in $\mathfrak{a}_2$ is an S_{a_2}-radical ring. Now if $R \in \mathfrak{a} \cap S_{a_2}$ then R is an S_{a_2}-radical ring so that $D(R/O)$ contains a non-zero ring which is isomorphic to a ring in $\bar{\mathfrak{a}}_2$. Then $D(R)$ containsa non-zero ring I which is isomorphic to a ring in $\bar{\mathfrak{a}}_2$, i.e., $I \approx A/K$ where $A \in \mathfrak{a}_2$ and $K \leq A$. Let n be the smallest positive integer such that I is in $D_n(R)$. Then there exists a finite set $\{J_i : i=1, 2. \cdots, n\}$ of rings with $J_i \in D_i(R)$ for $i=1, 2, \cdots, n$ and $J_{i+1} \leq J_i$ for $i=1, \cdots, n-1$ and $J_n = I$. Thus $O \neq I = J_n \leq J_{n-1} \leq \cdots \leq J_1 \leq R$. By (1) we have $J_1 \approx R$. From (3) and (1) it then follows that $I = J_n \approx J_{n-1} \approx \cdots \approx J_1 \approx R$, i.e., $I \approx R$ and so $R \approx A/K$. But $R \in \mathfrak{a}$ and $A \in \mathfrak{a}_2$ so that by (2) we have $A \approx A/K$; whence $R \approx A$ and $R \in \mathfrak{a}_2$. Hence, we see that every non-zero ring in $\mathfrak{a}_1$ is S_{a_2}-semi-simple.

By (1) every non-zero ideal of a ring in $\mathfrak{a}_1$ can be mapped homomorphically onto some non-zero rino in $\mathfrak{a}_1$, Hence we can construct the upper radical property S_{a_1} determined by $\mathfrak{a}_1$, and every ring in $\mathfrak{a}_1$ is S_{a_1}-semi-simple. Now let $O \neq R$ be in $\mathfrak{a}$ and be S_{a_1}-semi-simple. Then there exists a non-zero ring A in $\mathfrak{a}_1$ and an ideal I of R such that $R/I \cong A$. But $R \in \mathfrak{a}$. and $A \in \mathfrak{a}_1$ Therefore, by (2), $R/I \approx R$ and so $R \approx A$, i.e., $R \in \mathfrak{a}_1$. Thus every non-zero ring in $\mathfrak{a}_2$ must be an S_{a_1}-radical ring. This completes the proof of the theorem.

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