HOMEOMORPHISMS ON MANIFOLDS

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Let $H(X)$ be the class of all homeomorphisms of a topological space X onto itself. If X is an n-manifold, then X is a strong local homogeneous $(S.L.H.)$, i.e., for every neighborhood of any point x, there exists a subneighborhood $U(x)$ such that for any $z \in U(x)$ there exists g in $H(X)$ with $g(x)=z$ and with g equal to the identity on the complement of $U(x)$. However there exist $S.L.H.$ spaces which are not n-manifolds, for example, the zero-dimensional completely regular spaces [1], the universal curve [2] and the normed linear spaces [1, 3]. Therefore being a $S.L.H.$ space does not characterize an n-manifold. Since $S.L.H.$ is defined by the existence of one homeomorphism and moving one point onto another within a small open set, we intend to formulate a similar concept, the existence of a finite family of homeomorphisms which are the identity map outside a small open set and move a set to satisfy certain relations. A topological space X is called finitely complementary $(F.C.)$ if for every neighborhood U of any point x and any open set V such that $x \in \mathrm{Bndry}(V)$ there exists a finite subfamily $\{f_1, \cdots, f_n\}$ of $H(X)$ such that $\cup\{f_i(V): i=1, 2\cdots, n\} \cup \{x\}$ is an open set and each f_i is the identity map at x and outside U. The purpose of this paper is to prove that every finite-dimensional manifold is $L.F.C.$ This proposition is very useful in studying $H(X)$ [4]. Also we raise many interesting questions about $L.F.C.$

LEMMA 1. *A normed linear space is $S.L.H.$*

PROOF. See [1] or [3].

LEMMA 2. *Let U be a unit open ball with center 0 in Euclidean n-space E^n and V be an open set such that $0 \in \mathrm{Bndry}(V)$. Let L be a line segment with 0 as one end-point. Then there exists $f \in H(E^n)$ such that F is the identity at 0 and outside U and $0 \in U(L \cap f(V))$.*

PROOF. Since $0 \in$ Bndry (V), there exists a sequence of points $\{p_i\}_{i=1}^{\infty}$ in V such that $\{p_i\}_{i=1}^{\infty}$ converges to 0. Without loss of generality we may assume that the distances $d(0, p_i)$ are strictly decreasing and $p_1 \in L$. Let q_2 be the point in L such that $d(q_2, 0)=d(p_2, 0)$. Then there is an arc $[q_2, p_2]$ in the sphere with center 0 and radius $d(q_2, 0)$. Associate with each point q in $[q_2, p_2]$ an open **ball** B_q such that $Cl(B_q) \cap \cup\{p_i : i \neq 2\} = \phi$. Since $[q_2, p_2]$ is compact, there exists a subcover $\{B_1, \cdots, B_m\}$ such that $p_2 \in B_1$, $q_2 \in B_n$ and $B_i \cap B_{i+1} \neq \phi$, $i=1, 2, \cdots, m-1$. Let $x_i \in B_i \cap B_{i+1}$ for each $i=1, 2, \cdots, m-1$. By LEMMA 1, there exists $f_i \in H(E^n)$, $i=1, 2, \cdots, m$ such that f_i is the identity at 0 and outside B_i and $f_1(p_2)=x_1$, $f_k(x_{k-1})=x_k$ for $2 \leq k \leq m-1$ and $f_m(x_{m-1})=q_2$. Let $F_2=f_m \cdot f_{m-1} \cdots f_2 \cdot f_1$. Then F_2 is the identity at 0 and outside $\bigcup_{i=1}^{m} B_i = V_2$ and $F_2(p_2)=q_2$. Similarly we pick $q_i \in L$ and open sets V_i such that $V_i \cap V_j = \phi$ for $i \neq j$, and $F_i \in H(E^n)$ such that F_i is the identity at 0 and outside V_i and $F_i(p_i)=q_i$. Let F be the identity on $E^n \setminus \bigcup_{i=2}^{\infty} V_i$ and $F=F_i$ on V_i, $i=2, 3, \cdots$. It is now clear that $F \in H(E^n)$, $F(p_i)=q_i$. each q_i lies on L and $0 \in Cl(L \cap f(V))$.

Let $p=(x_1, x_2, \cdots, x_n)=(\gamma, \theta_2, \cdots, \theta_n)$ be any point different from 0 in E^n where $(\gamma, \theta_2, \cdots, \theta_n)$ is defined as follows:

$$\gamma=\sqrt{x_1^2+\cdots+x_n^2}$$

$$\sin \theta_j=\frac{x_j}{\sqrt{x_1^2+\cdots+x_j^2}}, \qquad 3 \leq j \leq n \text{ and } -\frac{\pi}{2}<\theta_j<\frac{\pi}{2}$$

$$\cos \theta_2=\frac{x_2}{\sqrt{x_1^2+x_2^2}}, \ \sin \theta_2=\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \qquad -\pi<\theta<\pi.$$

Thus we have

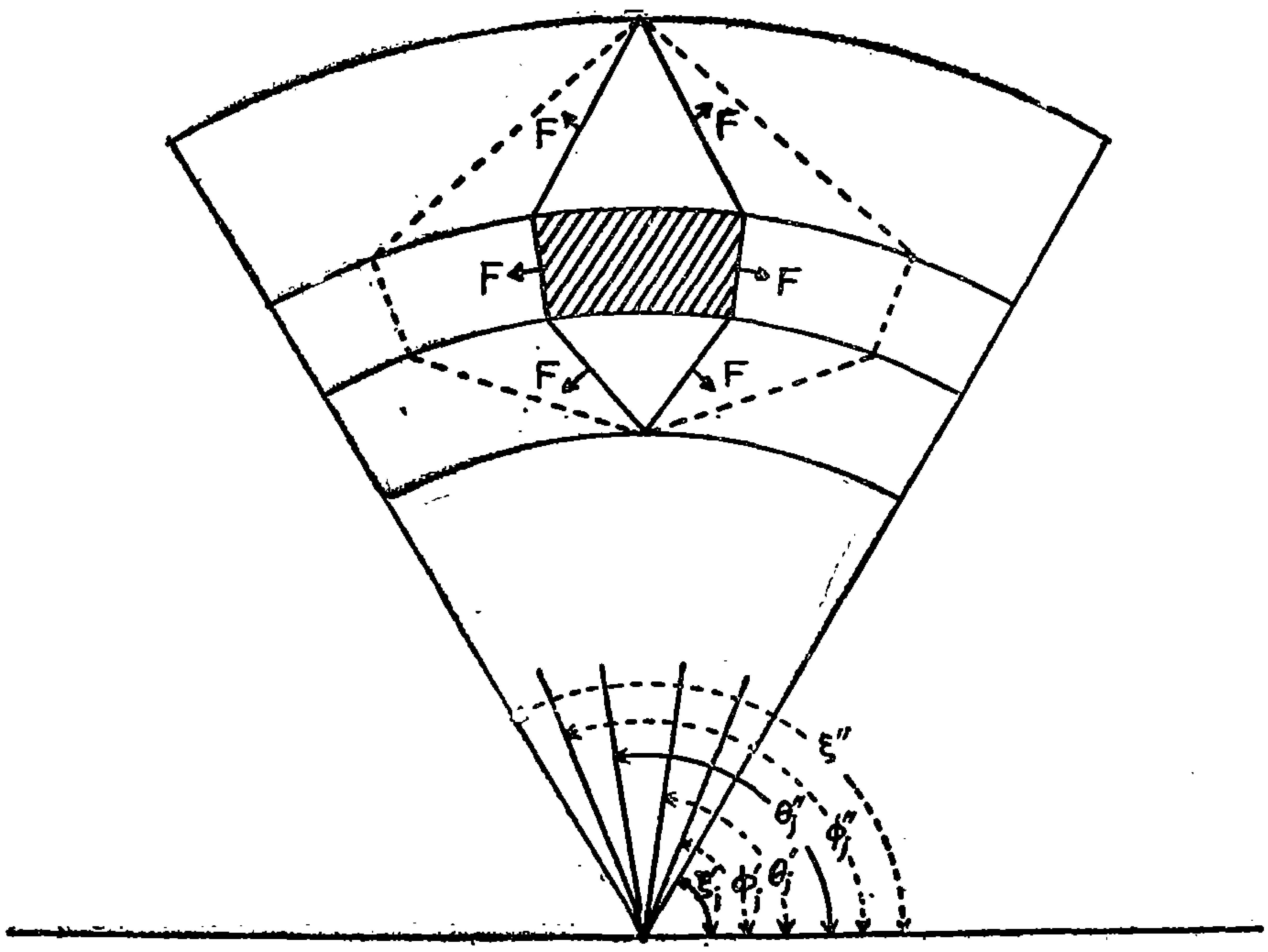
$$x_1=\gamma \cos\theta_n \cdots \cos\theta_3 \cos\theta_2,$$
$$x_2=\gamma \cos\theta_n \cdots \cos\theta_4 \cos\theta_3 \sin\theta_2,$$
$$x_3=\gamma \cos\theta_n \cos\theta_{n-1} \cdots \cos\theta_4 \sin\theta_3,$$
$$\cdots\cdots\cdots$$
$$x_{n-1}=\gamma \cos\theta_n \sin\theta_{n-1},$$
$$x_n=\gamma \sin\theta_n.$$

This defines a homeomorphism from the product space $(0,\infty)\times(-\pi,\pi)\times\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^{n-2}$ onto $E^n \setminus B$ where $B=\{(x_1, \cdots, x_n) \in E^n \mid \text{there exists an } i \leq n \text{ such that } x_j=0 \text{ for all } j \leq i\}$. Thus we have the following lemma.

LEMMA 3. *The family of all sets of the form* $V(\gamma', \gamma''; \theta_i', \theta_i''; i=2, 3, \cdots, n) = \{(\gamma; \theta_2, \cdots, \theta_n): \gamma_1<\gamma<\gamma_2, \theta_i'<\theta_i<\theta_i'', i=2, \cdots, n\}$ *where* $-\pi<\theta_2'<\theta_2''<\pi$, $-\frac{\pi}{2}<\theta_i'<\theta_i''<\frac{\pi}{2}$, $i=3, 4, \cdots, n$ *form a basis for* $E^n \setminus B$ *with the relative topology.*

LEMMA 4. *For each* $V(\gamma', \gamma''; \theta_i', \theta''_i; i=2, 3, \cdots, n)$ *and* $0<s<\gamma'<\gamma''<t$, $-\pi<\xi_2'<\varphi_2'\leq\theta_2'<\theta_2''\leq\varphi_2''<\xi_2''<\pi$, $-\frac{\pi}{2}<\xi_j'<\varphi_j'\leq\theta_j'<\theta_j''\leq\varphi_j''<\xi_j''<\frac{\pi}{2}$, $3\leq j\leq n$ *there exists a homeomorphism* F *of* E^n *onto itself such that* $F(x)=x$ *for* $\|x\|\leq s$ *or* $\|x\|\geq t$ *or* $x\in B$ *or* $x=(\gamma, \theta_2, \cdots, \theta_n)$ *with* $\theta_j<\xi_j'$ *or* $\theta_j>\xi_j''$ *for some* j *and* $F(V)=\{(\gamma, \theta_2, \cdots, \theta_n): \gamma'<\gamma<\gamma'', \varphi_j'<\theta_j<\varphi_j''\}$

PROOF. This can be easily seen by the following picture.



LEMMA 5. *Let* $V=\{x\in E^n: 0<\gamma_1<\|x\|<\gamma_2\}$, *and let* a_1, a_2 *be any two numbers such that* $0<a_1<a_2<\gamma_1$. *Then there is a homeomorphism* F *from* E^n *onto* E^n *such that* $F(x)=x$ *when* $\|x\|\geq\gamma_2$ *or* $\|x\|\leq a_1$ *and* $F(V)=\{x\in E^n : a_2<\|x\|<\gamma_2\}$.

PROOF. This is clear.

LEMMA 6. *Let* U *be the open unit ball in* E^n *with center* p_0 *and* $A\subset U$ *be such that* $p_0\in Cl\,(\mathrm{Int}(A))-A$. *Then there exist homeomorphisms* G_i, $i=1, 2, \cdots, n$ *of* E^n *onto itself such that*

$$\{x=(\gamma, \theta_2, \cdots, \theta_n): 0<\gamma<k, -\frac{\pi}{2}<\theta_i'<\theta_i<\theta_i''<\frac{\pi}{2}, i=3, 4, \cdots, n, -\pi<\theta_3'<\theta_3<\theta_3''<\pi\}$$

$\subset\bigcup_{i=1}^{m} G_i(\mathrm{Int}\,(A))$ *for some* k *and* θ_i', θ_i'', $i=2, 3, \cdots, n$, *and* $G_i(p_0)=p_0$ *and* G_i

is the identity in E^n-U *for each* i.

PROOF. By LEMMA 2, there exists a homeomorphism F_1 of E^n onto itself such that $p_0 \in Cl(R \cap F_1(\text{Int}(A)))$ where R is the ray $\left\{(x: x=0 \text{ or } x=(\gamma, \theta_2, \cdots, \theta_n), \theta_2 =\theta_3=\cdots=\theta_n=\frac{\pi}{4}\right\}$. Let $\{p_i\}_{i=1}^{\infty} \subset F_1(\text{Int}(A)) \cap R$ such that $\{p_i\}_{i=1}^{\infty}$ converges to p_0 and the distances $\{d(p_i, p_0)\}_{i=1}^{\infty}$ are strictly decreasing. By LEMMA 3, for each p_j, there is a V_j of the form $V_j=\left\{\{(\gamma, \theta_2, \cdots, \theta_n): \gamma_j'<\gamma<\gamma_j'', \frac{\pi}{8}<\theta_{ji}'<\theta_i <\theta_{ji}''<\frac{3\pi}{8}, i=2, 3, \cdots, n\right\}$ such that $p_j \in V_j \subset \text{Int}(A)$ and $\gamma_1''>\gamma_1'>\gamma_2''>\gamma_2' >\cdots$, $j=1, 2, \cdots$. By LEMMA 4, there exists a sequence of homeomorphisms $\{f_i\}_{i=1}^{\infty}$ of E^n onto itself such that $f_i(x)=x$ when $\|x\|\geq S_i$ or $\|x\|\leq S_{i-1}$ where $\gamma_i''<S_i <\gamma_{i-1}'$ or $x=(\gamma, \theta_2, \cdots, \theta_n)$, $\theta_j\leq\frac{\pi}{16}$ or $\theta_j\geq\frac{7\pi}{8}$ for some j and $f_i(V_i)=\left\{(\gamma, \theta_2, \cdots, \theta_n): \gamma_i''<\gamma<\gamma_i'', \frac{\pi}{8}<\theta_j<\frac{3\pi}{8}, j=2, \cdots, n\right\}$. Define F_2 on E^n as follow:

$$F_2(x)=x \text{ when } \|x\|>S_1$$

$$F_2(x)=f_2(x) \text{ when } S_{i-1} \leq\|x\|\leq S_i.$$

Then F_2 is a homeomorphism of E^n onto itself such that $F_2(F_1(\text{Int}(A))) \supset\left\{(\gamma, \theta_2, \cdots, \theta_n): \gamma_j'<\gamma<\gamma_j'' \text{ for some } j \text{ and } \frac{\pi}{8}<\theta_i<\frac{3\pi}{8}\right\}$, and $F_2(p_0)=p_0$. Pick t_i', t_i'' for each i such that $\gamma_i'<t_i'<t_i''<\gamma_i''$, $i=1, 2, \cdots\cdots$. Then by LEMMA 5, there exists a sequence of homeomorphisms $\{g_i\}_{i=1}^{\infty}$ of E^n onto itself such that g_i is fixed when $\|x\|\geq\gamma_i''$ or $\|x\|\leq t_{i+1}'$ and g_i maps the set $\{x: \gamma_i'<\|x\|<\gamma_i''\}$ onto $\{x: t_{i+1}''<\|x\|<\gamma_i''\}$. Let F_3 and F_4 be defined as follows:

$$F_3(x)=f_{2i+1}(x) \text{ when } x\in\{x: \gamma_{2i+2}<\|x\|<\gamma_{2i+1}''\}, \ i=0, 1, 2, \cdots,$$
$$F_3(x)=x \text{ otherwise},$$
$$F_4(x)=f_{2i}(x) \text{ when } x\in\{x: \gamma_{2i+1}'<\|x\|<\gamma_{2i}''\}, \ i=1, 2, \cdots,$$
$$F_4(x)=x \text{ otherwise}.$$

Then F_3, F_4 are both homeomorphisms of E^n onto E^n and $F_3(p_0)=F_4(p_0) =p_0$. Let $G_1=F_3F_2F_1$ and $G_2=F_4F_2F_1$. Then we have $\{x=(\gamma, \theta_2, \cdots, \theta_n): 0<\gamma <\gamma_1'', \frac{\pi}{8}<\theta_i<\frac{3\pi}{8}, i=2, \cdots, n\} \subset \bigcup_{i=1}^{2} G_i(\text{Int}(A))$ and $G_i(p_0)=p_0$, $i=1, 2, \cdots$ and G_i is the identity in E^n-U. By LEMMA 6, if $p_0\in Cl(\text{Int}(A))\subset U$, and $x\notin B$, then there exists an open cone C_x of height $k<1$ and a finite family of homeomorphisms $f_{x_1},\cdots,f_{x_m}$ of E^n onto E^n each of which is fixed outside U and p_0 such that $C_x\subset\bigcup_{i=1}^{n} f_{x_i}(\text{Int }(A))$. Since we can choose another coordinate system, we can do the same thing for $x\in B$. Thus associate each x in U with norm $k/2$, with a cone C_x. Thus $\{C_x\colon \|x\| =k/2\}$ forms a cover for the $(n\text{-}1)$-sphere with center p_0 and radius $k/2$ and hence there exists a finite set of numbers, $\{x_1, x_2, \cdots, x_l\}$ such that $C_{x_1}, \cdots, C_{x_l}$ form a cover for $\{x\colon \|x\|=k/2\}$. Since each cone has the property that $x\in C$ implies $(p_0, x)\subset C$, then $C_{x_1}, \cdots, C_{x_l}$ form a cover of $\{x\colon 0<\|x\|\le k/2\}$. Hence there exists a finite family, $\{f_i\}_{i=1}^{n}$ such that $\{x\colon 0<\|x\|\le k/2\}\subset\bigcup_{i=1}^{m} f_i(\text{Int}(A))$. Thus we have the following lemma.

LEMMA 7. *Let U be an open unit ball in E^n with center p_0 and let A be a subset of U such that $p_0\in Cl(\text{Int}(A))-A$. Then there exists a finite family of homeomorphisms $\{f_i\}_{i=1}^{m}$ of E^n onto itself such that $\{x\colon 0<\|x\|<\gamma\}\subset\bigcup_{i=1}^{m} f_i(\text{Int }(A))$ where $\gamma<1$ and f_i is fixed at p_0 and outside U for all i.*

From LEMMA 7, we immediately have the desired theorem that every n-manifold is finitely complementary.

The following questions might be interesting.

1. Is every Hilbert space or Banach space finitely complementary?
2. Do *S.L.H.* and *F.C.* imply locally Euclidean?
3. Are zero-dimensional completely regular spaces and the universal curves finitely complementary?
4. Is every homogeneous *F.C.* space *S.L.H*?

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