ON PSEUDO MANIFOLD WITH $(fr,\ g)$-STRUCTURE.

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Yano [4] introduced the concept of f-structure satisfying $f^3+f=0$ on an n-dimensional differentiable manifold, and the author inverstigated integrability conditions from the global view point.

The f-structure may be regarded as a generalization of the almost complex structure and almost contact structure. Later on Kodo [1] has defined a normal $(fr,\ g)$-structure and inverstigated such an infinitesimal transformation v^h of a differentiable manifold with f-structure as leaves the structure tensor f_j^i invariant.

In the preset paper, we shall define a pseudo-manifold with $(fr,\ g)$ structure and prove some identities valid in this manifold.

Furthmore, we define a f_j^i-invariant vector which is a generalization of a analytic vector in almost complex spaces, and deduce some theorems in pseudo manifold with (fr, g)-structure.

§ 1. Indroduction.

We consider an n-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhood $\{x^h\}$, and a tensor field f_i^h of type (1, 1) and of class C^∞ satisfying

$$f_i^t f_t^s f_s^h + f_i^h = 0. \qquad (i, h, s, t, \cdots\cdots = 1, 2, \cdots\cdots, n) \tag{1. 1}$$

For a tensor f_i^h satisfying (1. 1) the operators

$$l_i^h = -f_i^t f_t^h \qquad \text{and} \qquad m_i^h = -l_i^h + \delta_i^h \tag{1. 2}$$

applied to the tangent space at a point of the manifold are complementary projection operators. Thus if there is given a tensor f_i^h satisfying (1. 1), there exist complementary distributions L and M corresponding to the projection operators l_i^h and m_i^h respectively. If the rank of f is r, then we call such a structure an $f$structure of rank r $(r \leqq n)$, in this case the dimensions of L and M are r and $n-r$ respectively.

For f_i^h satisfying (1. 1) and l_i^h, m_i^h defined by (1. 2), we have

$$\text{(1. 3)}\qquad \begin{cases} l_i^t f_t^h = f_t l^h = f_i^h, & l_i^t l_t^h = l_i^h, \\ m_i^t m_t^h = m_i^h, & f_i^t m_t^h = l_i^t m_t^h = 0. \end{cases}$$

If the rank of f is n, then $l_i^h = \delta_i^h$ and $m_i^h = 0$, so that, we find that the f-structure of rank n is an almost complex structure. And if the rank of f is n-1, then the distribution M is one dimensional and m_j^i should have the form:

$$\text{(1. 4)}\qquad m_j^i = f^i f_j.$$

where f^i and f_j are contravariant and covariant vector respectively. From the relations (1.2) we have

$$\text{(1. 5)}\qquad f_j^t f_t^i = -f_j^i + f^i f_j.$$

Therefore, we find that the f-structure of rank $n-1$ is an almost contact structure defined by Sasaki [4].

It is well known [6] that a manifold with f-structure of rank r always admits a positive definite Riemannian metric tensor g_{ji} such that

$$\text{(1. 6)}\qquad f_j^t f_i^s g_{ts} = g_{ji} - m_{ji}, \qquad \text{where} \qquad m_{ji} = m_j^t g_{ti},$$

from which we see that the tensor m_{ji} is a symmetric one.

If an f-structure of rank r admits a positive definite Riemannian metric defined by (1.6), then we shall call the structure an (f_r, g)-structure.

Next, operating ∇_j to (1.1), we find

$$\text{(1. 7)}\qquad f_i^s f_s^h \nabla_j f_i^t + f_i^t f_s^h \nabla_j f_t^s + f_i^t f_t^s \nabla_j f_s^h + \nabla_j f_i^h = 0,$$

where ∇_j denotes the operator of covariant derivative with respect to the Riemannian connection formed with g_{ji}, and tensor $f_{ji} = f_j^t g_{ti}$ is a skew-symmetric one [1].

Applying ∇_h to (1.6), we have

$$\text{(1. 8)}\qquad \nabla_h m_{ji} + f_{jt} \nabla_h f_i^t + f_{it} \nabla_h f_j^t = 0.$$

From (1.3) and (1.7), we find [2]

$$\text{(1. 9)}\qquad f_i^t l_h^s \nabla_j f_{ts} = l_i^t f_h^s \nabla_j f_{ts}.$$

By virtue of (1.2) and (1.8), we have

$$\text{(1.10)}\qquad m_t^j m_s^i \nabla_h m_{ji} = 0.$$

§ 2. Pseudo-manifold with (fr, g)-structure.

In a manifold with (f_r, g)-structure, if its structure tensor f_i^h satisfies

$$N_{ji}^h \equiv f_j^l \nabla_l f_i^h - f_i^l \nabla_l f_j^h - (\nabla_j f_i^l - \nabla_i f_j^l) f_l^h = 0. \tag{2.1}$$

and

$$F_{jih} \equiv \nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji} = 0, \tag{2.2}$$

$$M_{jih} \equiv \nabla_j m_{ih} + \nabla_i m_{hj} + \nabla_h m_{ji} = 0, \tag{2.3}$$

where N_{ji}^h is the Nijenhuis tensor for the structure tensor f_i^h defined by Yano [4], then we shall call the manifold a pseudo-manifold with (f_r, g)-structure.

Transvecting (2.1) with m_h^k, we have

$$f_j^l f_i^h (\nabla_l m_h^k - \nabla_h m_l^k) = 0,$$

which is equivalent to

$$l_j^l l_i^h (\nabla_l m_h^k - \nabla_h m l^k) = 0. \tag{2.4}$$

Thus, if the Nijenhuis tensor vanishes, then the distribution L is integrable [4].

Next, the Nijenhuis tensor can be written as

$$N_{jih} \equiv N_{ji}^k g_{kh} = f_j^l F_{lih} - f_i^l F_{ljh} + \nabla_i m_{hj} - \nabla_j m_{ih} - \nabla_h m_{ji} - 2 f_i^l \nabla_h f_{jl}.$$

Using of (2.1) and (2.2), we get

$$\nabla_i m_{hj} - \nabla_j m_{ih} - \nabla_h m_{ji} = 2 f_i^l \nabla_h f_{jl}. \tag{2.5}$$

Transvection (2.5) with $m_t^j m_s^j$ by virtue of (1.10) we have

$$m_t^j m_s^i (\nabla_i m_j^h - \nabla_j m_i^h) = 0. \tag{2.6}$$

This equation shows that the distribution M is integrable [1]. Thus we have

THEOREM 1. *In a pseudo-manifold with (f_r, g)-structure, the distributions L and M are integrable.*

From (2.3) and (2.5), we get

$$\nabla_i m_{hj} = f_i^l \nabla_h f_{jl}. \tag{2.7}$$

Since the tensor m_{hj} is symmetric in h and j, we find

(2. 8) $$f_i^l \nabla_h f_{jl} - f_i^l \nabla_j f_{hl} = .$$

taking account of (2, 2), we get

(2. 9) $$f_i^l \nabla_l f_{jh} = 0,$$

which is equivalent to

(2.10) $$l_i^l \nabla_l f_{jh} = 0.$$

Transvecting (2.9) with f_k^h, using of (2. 7) we get

$$f_i^l \nabla_k m_{lj} = 0,$$

from which taking account of (1.3), we have

(2.11) $$m_j^l \nabla_k f_{il} = 0.$$

From (2.2), we find

$$m_j^l \nabla_l f_{ik} = 0,$$

which is equivatent to

(2.12) $$m_i^l \nabla_l f_{jh} = 0.$$

From (2.10) and (2.12), we have

(2.13) $$\nabla_i f_{jh} = 0.$$

Conversely, if, in a manifold with (fr, g)-structure, (2.13) holds good, then we get $N_{.i\cdot}=0$, $F_{jih}=0$ and $M_{jih}=0$, which shows that the manifold is a pseudo-manifold. Thus we have

THEOREM 2. *A necessary and sufficient condition the manifold with (fr, g)-structure is pseudo-manifold is that* (2.13) *holds good.*

In a manifold with (fr, g)-structure, if affine connection Γ^i_{jk} satisfies

(2.14) $$\nabla_i f_{jk} = \partial_i f_{jk} - f_{ak}\Gamma^a_{ij} - f_{ja}\Gamma^a_{ik} = 0,$$

then we shall call the affine symmetic connection Γ^i_{jk} is an f-connection defined by Yano.

Thus we have

THEOREM 3. *In a pseudo-manifold with (fr, g)-structure, an affine symmetric connection Γ^i_{jk} is an f-connection.*

§ 3, Curvatures.

In this section, we shall assume we are in a pseudo-manifold with (fr, g)-structure.

Let $K_{hji}{}^{l}$ be the Riemannian curvature tensor, and we put

(3. 1) $$K_{ji}=K_{lji}{}^{l}, \qquad K=g^{ji}K_{ji}, \qquad K_{hjil}=K_{hji}{}^{m}\, g_{ml},$$

(3. 2) $$H_{ji}=f^{lh}K_{hjil}, \qquad H=-f^{ji}H_{ji}.$$

Applying the Ricci formulae to f_i^h, we have the following identities which are valid in a pseudo-manifold

(3. 3) $$\nabla_k\nabla_j f_i^h-\nabla_j\nabla_k f_i^h=K_{kjl}{}^{h}f_i^l-K_{kji}{}^{l}f_l^h.$$

Taking account of (2. 13), we get

(3. 4) $$K_{kjl}{}^{h}\, f_i^l-K_{kji}{}^{l}f_l^h=0.$$

Transvecting (3. 4) with g^{ji}, we get

$$K_k^l f_l^h=K_{kjl}{}^{h}f^{jl}=\frac{1}{2}(K_{kjl}{}^{h}-K_{klj}{}^{h})f^{jl},$$

from which we have

(3. 5) $$K_k^l f_{lh}=-\frac{1}{2}K_{jlkh}f^{jl}.$$

From (3. 2), we get

(3. 6) $$K_k^l f_{lh}=-H_{kh}.$$

Since the tensor H_{kh} is a skew symmetric in k and h, we get

(3. 7) $$K_k^l f_l^h=K_l^h f_k^l.$$

Transvecting (3. 6) with f^{kh}, we have

(3. 8) $$K-H=m^{kh}K_{kh}.$$

Next, we consider a pseudo-manifold of positive constant curvature, in this case the Riemannian curvature tensor takes the form;

(3. 9) $$K_{hjil}=\frac{K}{n(n-1)}(g_{hl}g_{ji}-g_{jl}g_{hi}).$$

Transvecting (3.7) with g^{lh}, we get

(3.10) $$K_{ji}=\frac{K}{n}g_{ji},$$

from which we have

$$m^{ji}K_{ji}=\frac{1}{n}m_a^a K.$$

Taking a ccount of (3.8), we have

(3.11) $$K-H=\frac{1}{n}m_a^a K.$$

Transvecting (3.9) with $f^{hl}f^{ji}$ and using of $f^{hl}f^{ji}g_{hl}g_{ji}=0$, we have

(3.12) $$H=\frac{K}{n(n-1)}l_a^a.$$

which is equivalent to

(3.13) $$K-H=K\left(1-\frac{l_a^a}{n(n-1)}\right),$$

From (3.10) and (3.12), we have

(3.14) $$\frac{n-2}{n-1}K\left(1-\frac{m_a^a}{n}\right)=0.$$

Since the relation $m_a^a \neq n$, we have $K=0$, $(n>2)$. Thus we have

THEOREM 4. *In a pseudo-manifold with (fr, g)-structure, there does not exist a manifold of non-zero constant curvature.*

§ 4. f_j^i-invariant.

In this section, we shall consider in a pseudo-manifold with (fr, g)-structure a vector field v^h satisfying

(4. 1) $$\pounds_v f_j^i \equiv -f_j^l\nabla_l v^i+f_l^i\nabla_j v^l=0,$$

where $\pounds_v$ denotes the Lie derivative with respect to an infinitesimal transformation v^h, then we shall call the vector field v^h is f_j^i-invariant. From (1. 2), we easily get

(4. 2) $$\pounds_v m_j^i=0.$$

And the following identities are well known:

(4. 3) $$\pounds_v\nabla_h f_j^i-\nabla_h\pounds_v f_j^i=f_j^l\pounds_v\{{}_{hl}^{\,i}\}-f_l^i\pounds_v\{{}_{hj}^{\,l}\},$$

$$(4.4)\qquad \pounds_v\{{}^{\,i}_{hj}\}=\frac{1}{2}g^{il}[\nabla_h\pounds_v g_{lj}+\nabla_j\pounds_v g_{hl}-\nabla_l\pounds_v\, g_{hj}].$$

In this manifold, by virtue of (2.13), we have

$$f_j^l\pounds_v\{{}^{\,i}_{hl}\}-f_l^i\pounds_v\{{}^{\,l}_{hj}\}=0.$$

Contracting for i and h, we get

$$(4.5)\qquad f_j^l\pounds_v\{{}^{\,t}_{tl}\}-f_l^t\pounds_v\{{}^{\,l}_{tj}\}=0.$$

On the other hand, from (4.2) we have

$$(4.6)\qquad m_j^l\pounds_v\{{}^{\,t}_{tl}\}-m_l^t\pounds_v\{{}^{\,l}_{tj}\}=0.$$

Substituting (4.4) in (4.5) and (4.6) respectively, we obtain

$$(4.7)\qquad \frac{1}{2}f_j^l g^{ts}(\nabla_l\pounds_v g_{ts})+f^{lt}(\nabla_l\pounds_v g_{tj})=0.$$

$$(4.8)\qquad m_j^l g^{ts}(\nabla_l\pounds_v g_{ts})-m^{ts}(\nabla_j\pounds_v g_{ts})=0.$$

In an n-dimensional Riemannian space, if a vector fied v^h satisfies each of the following conditions;

$$(4.9)\qquad \pounds_v g_{ji}\equiv\nabla_j v_i+\nabla_i v_j=0,$$

$$(4.10)\qquad \pounds_v g_{ji}\equiv\nabla_j v_i+\nabla_i v_j=2\phi g_{ji},$$

$$(4.11)\qquad \pounds_v\{{}^{\,h}_{ji}\}\equiv\nabla_j\nabla_i v^h+K_{ji}{}^{k}{}_{l}\,v^l=\delta_j^h\Psi_i+\delta_i^h\Psi_j.$$

then it is called a Killing vector, a conformal Killing vector and a projective Killing vector, respectively, where

$$\phi=\frac{1}{n}\nabla_l v^l,\qquad\qquad \Psi_i=\frac{1}{n+1}\nabla_i\nabla_l v^l.$$

If a conformal Killing vector v^h is at the same time f_j^i-invariant, then substituting (4.10) in (4.7) and (4.8) respectively, we find

$$(4.12)\qquad f_j^l\phi_l=0,$$

and

$$(4.13)\qquad nm_j^l\phi_l-m_l^l\phi_j=0.\qquad\qquad (n>2)$$

(4.12) is equivalent to

$$(4.14)\qquad l_j^l\phi_l=0.$$

From (4.12) and (4.13) and by virtue of (1.2), we have

$$\left(1-\frac{1}{n}m_l^l\right)\phi_j=0,$$

from which we have

(4.15) $$\phi_j=0,$$

which is equivalent to

(4.16) $$\nabla_j\nabla_l v^l=0.$$

As the manifold is compact, using the Green's theorem, we deduce

(4.17) $$\nabla_l v^l=0.$$

Thus we have

THEOREM 5. *In a compact pseudo-manifold with (fr, g)-structure, a conformal Killing vector v^h which admits f_j^i-invariant is a Killing vector.*

For a projective Killing vector v^h which is at the same time f_j^i-invariant, substituting (4.11) in (4.5) and (4.6) respectively, we get $\nabla_i\nabla_l v^l=0$, therefore, as the space is compact, we have $\nabla_l v^l=0$, that is, the vector becomes a Killing one.

Thus we have

THEOREM 6. *In a compact pseudo manifold with (fr, g)-structure, a projective Killing vector v^h which admits f_j^i-invariant is a Killing vector.*

Next, applying ∇_k to (4.1) anp transvecting with g^{kj}, we get

$$f_l^i g^{kj}\nabla_k\nabla_j v-\frac{1}{2}f^{kj}(\nabla_k\nabla_j v^i-\nabla_j\nabla_k v^i)=0,$$

$$f_l^i g^{kj}\nabla_k\nabla_j v-\frac{1}{2}f^{kj}K_{kjl}{}^i v^l=0.$$

From (3.5) and (3.7), we have

(4.18) $$f_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^l v^i)=0.$$

which is equivalent to

(4.19) $$l_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^l v^j)=0.$$

Now, if a contravariant vector field v^k is orthogonal to the distribution M, then we have

(4.20) $$m_l^i v^l=0.$$

Applying the Ricci formula to m_l^i and taking account of (2.13), we have

$$(4.21) \qquad K_{kjl}{}^h m_i^l - K_{kji}{}^l m_l^h = 0,$$

transvecting (4.21) with g^{ji}, we get

$$(4.22) \qquad K_k^l m_l^h = K_{kjl}{}^h m^{jl},$$

since the tensor m_{jl} is a symmetric in j and l, we have

$$(4.23) \qquad K_k^l m_l^h = K_l^h m_k^l.$$

From (4.20) and (4.23), we have

$$(4.24) \qquad m_l^i (g_{kj} \nabla_k \nabla_j v^l + K_j^l v^j) = 0.$$

From (4.19) and (4.24), we find

$$(4.25) \qquad g^{kj} \nabla_k \nabla_j + K_j^l v^j = 0.$$

Thus we have

THEOREM 7. *In a compact orientable pseudo manifold with (fr, g)-structure, if an f_j^i -invariant vector field v^h admits $m_l^i v^l = 0$ and $\nabla_l v^l = 0$,* then it is a Killing vector.

§5. Integral formulae

Now, in a compact pseudo manifold with (f_r, g)-structure, we shall obtain a necessary and sufficient condition that a contravariant vector field v^i is f-invariant, For f-invariant vector v^i, we have

$$(5.1) \qquad f_l^i \nabla_j v^l - f_j^l \nabla_l v^i = 0.$$

Operating ∇_k to the last equation, we get

$$f_l^i \nabla_k \nabla_j v^l - f_j^l \nabla_k \nabla_l v^i = 0.$$

Transvecting this equation with g^{kj}, we find

$$(5.2) \qquad f_l^i g^{kj} \nabla_k \nabla_j v^l - \frac{1}{2} f^{kl} (\nabla_k \nabla_l v^i - \nabla_l \nabla_k v^i) = 0.$$

Applying the Ricci formula to v^i, we have the following identities which are valid in this manifold;

$$(5.3) \qquad \nabla_k \nabla_l v^i - \nabla_l \nabla_k v^i = K_{klj}{}^i v^j$$

From (3.5) and (5.3), (5.2) becomes

$$(5.4)\qquad f_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^i v^j)=0,$$

which is equivalent to

$$(5.5)\qquad l_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^i v^j)=0.$$

This equation is a necessary condition for a vector v^i to be f-invariant.

Next, we shall get a sufficient condition, For a contravariant vector v^i, if we put

$$T_{ji}\equiv(\underset{v}{\pounds} f_j^a)g_{ai}=-f_j^l\nabla_l v_i-f_i^l\nabla_l v_j,$$

then we have

$$(5.6)\qquad \frac{1}{2}T^2=(\nabla_j v_i)(\nabla^j v^i)-f_j^l f_i^m(\nabla_l v_m)(\nabla^j v^i)-\frac{1}{2}m_t^l\{(\nabla_l v_i)(\nabla^t v^i)+(\nabla_i v_l)(\nabla^i v^t)\}$$

and

$$(5.7)\qquad \nabla^j(T_{jl}f_i^l v^i)=(\nabla_j v_i)(\nabla^j v^i)-f_j^l f_i^m(\nabla_l v_m)(\nabla^j v^i)-m_t^l(\nabla_i v_l)(\nabla^i v^t)+l_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^l v^j)v_i.$$

From (5.6) and (5.7), we get

$$(5.8)\qquad \frac{1}{2}T^2-\nabla^j(T_{jl}f_i^l v^i)=-l_l^i(g^{kl}\nabla_k\nabla_j v^l+K_j^l v^j)v_i-\frac{1}{2}m_t^l\{(\nabla_l v_i)(\nabla^t v^i)-(\nabla_i v_l)(\nabla^i v^t)\}.$$

On the other hand, for a contravariant vector v^i, we have

$$(5.9)\qquad \underset{v}{\pounds} m_t^i\equiv -m_t^l\nabla_l v^i+m_l^i\nabla_t v^l,$$

from which we find

$$(5.10)\qquad \underset{v}{\pounds} m_t^i(\nabla^t v_i)=-m_t^l\{(\nabla_l v_i)(\nabla^t v^i)-(\nabla_i v_l)(\nabla^i v^t)\}.$$

If we put

$$U_{ji}=m_j^t(\nabla_t v_i),\quad V_{ji}=m_j^t(\nabla_i v_t),$$

then we have

$$(5.11)\qquad U^2=m_t^l(\nabla_l v_i)(\nabla^t v^i),\quad V^2=m_t^l(\nabla_i v_l)(\nabla^i v^t).$$

From (5.10) and (5.11), the equation (5.8) may be written

$$\frac{1}{2}T^2-\nabla^j(T_{jl}f_i^l v^i)=-l_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^l v^j)v_i-\frac{1}{2}(U^2-V^2). \tag{5.12}$$

Hence, applying the Green's theorem, we have

LEMMA 1. *In a compact pseudo manifold with* (f_r, g)*-structure, the integral formula*

$$\int_{Mn}\left\{\frac{1}{2}T^2+\frac{1}{2}(U^2-V^2)+l_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^l v^j)v_i\right\}d\sigma=0 \tag{5.13}$$

is valid for any vector field v^i*, where* $d\sigma$ *means the volume element of the manifold* M_n*, and* $T_{ji}=(\underset{v}{£}f_j^l)g_{li}$*,* $U_{ji}=m_j^t\nabla_t v_i$*,* $V_{ji}=m_j^t\nabla_i v_t$*.*

From this lemma, we have

THEOREM 7. *In a compact pseudo manifold with* (f, g)*-structure, a necessary and sufficient condition that a contravariant vector* v^i *is f-invariant vector is that it satisfies*

$$l_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^l v^j)=0 \text{ and } U^2=V^2.$$

Now, If a contravariant vector v^i is orthogonal to the distribution M, that is,

$$m_l^i v^l=0. \tag{5.14}$$

Operating ∇_j to the last equation, we get

$$m_l^i\nabla_j v^l=0,$$

from which we have $V_{ji}=0$.

On the other hand, transvecting (3.7) with f_h^i, we find

$$K_j^r m_r^i=K_r^i m_j^r. \tag{5.15}$$

From (5.14) and (5.15), we get

$$m_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^l v^j)=0, \tag{5.16}$$

from which we have

$$l_l^i(g^{kj}\nabla_k\nabla_j v^l+K_j^l v^j)=g^{kj}\nabla_k\nabla_j v^i+K_j^i v^j \tag{5.17}$$

by virtue of (1.2).

Thus we have

LEMMA 2. *In a compact pseudo manifold with (f_r, g)-structure, the integral formula*

$$\int_{M_n}\left\{\frac{1}{2}T^2+\frac{1}{2}U^2+(g^{kj}\nabla_k\nabla_j v^i+K_j^{\ i}v^j)v_i\right\}d\sigma=0. \tag{5.18}$$

is valid for any vector field v^i which is orthogonal to the distribution M, where $d\sigma$ means the volume element of the manifold M_n, and

$$T_{ji}=(\pounds f_j^l)g_{li},\ \ U_{ji}=m_j^{\ t}(\nabla_t v_i).$$

From this lemma, we have

THEOREM 8. *In a compact pseudo manifold with (f_r, g)-structure, a necessary and sufficient condition that a contravariant vector v^i which is orthogonal to the distribution M is f-invariant is that*

$$g^{kj}\nabla_k\nabla_j v^i+K_j^{\ i}v^j=0.$$

From this theorem, we have

THEOREM 9. *In a compact orientable pseudo manifold with (f_r, g)-structure, a necessary and sufficient condition that f-invariant vector v^i which is orthogonal to the distribution M is a Killing vector is that it satisfies $\nabla_l v^l=0$.*

6. Pseudo Einstein manifold.

In this section, we shall consider a compact pseudo Einstein manifold with (f_r, g)-structure and give a theorem of f- invariant vector v^i. This theorem corresponds to Matsushima's theorem in a compact Einstein Kaehlerian space.

Now, we shall consider a contarvariant vector v^i satisfying

$$\underset{v}{\pounds} f_j^{\ i}=0 \text{ and } m_h^{\ i}v^h=0, \tag{6.1}$$

from which we find

(6.2) $$\nabla^r \nabla_r v^h + K_r{}^h v^r = 0$$

by virtue of (5.19).

As M_n is an Einstein manifold, it holds that

(6.3) $$K_{ji} = c g_{ji}, \quad c = \frac{K}{n},$$

where $c \neq 0$.

Inthis case, (6.2) may be written the following

(5.4) $$\nabla^r \nabla_r v^h + c v^h = 0.$$

Operating ∇_h in the last equation, we have

(6.5) $$\nabla^r \nabla_r (\nabla_h v^h) + 2c(\nabla_h v^h) = 0.$$

If we put

(6.6) $$\phi = \nabla_h v^h$$

then te equation (6.5) is written as

(5.7) $$\nabla^r \nabla_r \phi + 2c\phi = 0.$$

If a scalar function ϕ is a solution of (6.7), the equation

(6.8) $$\nabla^r \nabla_r \nabla_i \phi + c \nabla_i \phi = 0$$

is valid for an f-invariant vector v^i in this manifold. Hence the gradient ∇_i of ϕ for an f-invariant vector v^i is also f-invariant vector.

Now, we put

(6.9) $$p^h = v^h + \frac{1}{2c} \nabla^h \phi,$$

then the vector p^h is an f-invariant vector.

Operating ∇_h to the equation (6.9) and taking account of (6.7), we have

(6.10) $$\nabla_h p^h = 0.$$

from (6.4) and (6.8), we get

(6.11) $$\nabla^r \nabla_r p^h + K_r{}^h p^r = 0.$$

Hence the vector p^h is a Killing vector.

Neot, we put

(6.12) $$q^h=f_l{}^h[\frac{1}{2c}\nabla^l\phi],$$

then we find

(6.13) $$m_h{}^iq^h=0.$$

Hence the vector q^h is an f-invariant and orthogonal to the distribution M. Operating ∇_h to the equation (6.12), we have [7]

(6.14) $$\nabla_hq^h=0.$$

Hence, by the theorem 9, the vector q^h is a Killing vector.

From (6.9), we have

(6.15) $$v^h=p^h-\frac{1}{2c}\nabla^h\phi.$$

Transvecting this equation with $l_h{}^i$ and taking account of (1.2) and (6.12), we get

$$v^i-m_h{}^iv^h=l_h{}^ip^h+f_h{}^iq^h.$$

From the second term of (6.1), the last equation may be written as

(6.16) $$v^i=l_h{}^ip^h+f_h{}^iq^h.$$

where p^h and q^h are Killing vectors.

If an f-invariant vector v^i which is orthogonal to the distribution M is represented in the following difference form;

(6.17) $$v^i=l_h{}^ip'^h+f_h{}^iq'^h.$$

From (6.16) and (6.17), we have

$$l_h^i(p^h-p'^h)+f_h^i(g^h-g'^h)=0.$$

Transvecting the last equation with f_i^l, we have

(6.18) $$f_h^l(p^h-p'^h)-(q^l-q'^l)=0$$

by virtue of (6.13).

Operating ∇_l the equation (6.18) and taking account of (6.14), we find

(6.19) $$f_h{}^l\nabla_l(p^h - p'^h) = 0,$$

from which we have

(6.20) $$p^h = p'^h.$$

From (6.18) and (6.20), we have

(6.21) $$q^h = q'^h.$$

Thus we have

THEOREM 10. *In a compact pseudo Einstein manifold with (f_r, g)-structure, an f-invariant vector v^i which is orthogonal to the distribution M is uniquely represented in the form;*

$$v^i = l_h{}^i p^h + f_h{}^i q^h,$$

where p^h and q^h are Killing vectors.

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