A CHARACTERIZATION OF PARACOMPACT SPACES BY THE FILTERS IN THEM

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1. Introduction

The concept of a paracompact space has been introduced in 1944 by Dieudonné [4] as a generalization of certain compact spaces. In his paper, it has been proved that the product of paracompact space and every compact space (Hausdorff) is normal and that the set of all neighborhoods of the diagonal is a uniformity for it. The compactness has already been characterized through the device of the filter's formation. The purpose of this thesis is also to find a way to characterize the paracompactness in the similar filter's formation. As it has been suggested in the 2nd statement of Dieudonné's paper, the paracompactness could be formulated in terms of uniform structures. Corson proved in [3] that a T_2-space X is paracompact iff X admits a uniformity under which every filter, satisfying a Cauchy-like condition, i.e., weakly Cauchy filter, has a cluster point. Based on Corson's contribution, the present thesis attempts to construct another characterization of paracompactness by filter's formation:

A Hausdorff space $(X,\mathscr{T})$ is paracompact iff every filter in X has a cluster point in $(X,\mathscr{T})$, whenever the filter has a cluster point in each pseudo-metric space (X,p) whose topology is weaker than the original topology.

In what follows, we shall first discuss some basic concepts and terminologies relating to the paracompact space, and these will pave the way for the further development of the present thesis.

DEFINITION 1. A topological space is *paracompact* iff it is Hausdorff and each open cover has an open locally finite refinement.

Since it is not hard to show that a Hausdorff space is regular if each open cover has an open locally finite refinement. Hence the usual definition of paracompact space in Kelley [5] specifies regular instead of Hausdorff.

DEFINITION 2. A filter $\mathscr{F}$ in a uniform space $(X,\mathscr{U})$ is *weakly Cauchy* if for every $U \in \mathscr{U}$ some filter stronger than $\mathscr{F}$ becomes U small. That is, there is a filter $\mathscr{H}_U$ $\mathscr{H}_U \supset \mathscr{F}$ and $H\times H\subset U$ for some $H\in\mathscr{H}_U$.

DEFINITION 3. A cover $\mathscr{U}$ of a topological space is called an *even cover* iff there is a neighborhood V of the diagonal in $X\times X$ such that $\{V[x]: x\in X\}$ refines $\mathscr{U}$.

This concept is derived from the Lebesgue's covering Lemma for a pseudo-metric and compact space.

PROPOSITION. *Let X be a topological space such that each open cover is even. If U is a neighborhood of the diagonal in $X\times X$, then there is a symmetric neighborhood V of the diagonal such that $V\circ V\subset U$.*

The proof of the above proposition is given in Kelley [5].

THEOREM. *If each open cover of T_1 (regular) space X is even, then the family of all neighborhoods of the diagonal is a uniformity for X.*

PROOF. By the above proposition, the family $\mathscr{U}$ of all neighborhoods of the diagonal is clearly a uniformity for X. For any $U\in\mathscr{U}$, and $x\in X$, there exists neighborhood N_x of x such that $N_x\times N_x\subset U$. The uniform topology is weaker than the original topology. Conversely, let N be a neighborhood of x with respect to the original topology. Since X is T_1 (regular), there exists a closed neighborhood F of x such that $x\in F\subset N$. Then $(X\sim F)\times(X\sim F)\cup(N\times N)$ is a neighborhood of Δ. N is a neighborhood of x with respect to the uniform topology.

Let's consider $X=\{a,b,c\}$ and $\mathscr{T}=\{\phi,X,\{b\},\{c\},\{b,c\}\}$. $(X,\mathscr{T})$ is a T_0-space such that each open cover is even. Otherwise the family of neighborhoods of Δ is not a uniformity for $(X,\mathscr{T})$, but indiscrete uniformity.

COROLLARY. *If X is paracompact, then the family of all neighborhoods of the diagonal is a uniformity for X.*

2. Theorem

THEOREM . *A Hausdorff space $(X,\mathscr{T})$ is paracompact iff every filter in X has a cluster point in $(X,\mathscr{T})$, whenever the filter has a cluster point in each pseudo-metric space (X,p) whose topology is weaker than the original topology.*

LEMMA 1. *If X is paracompact, then the set of all neighborhoods of the diagonal is a uniformity for X, and the product of X and every compact (Hausdorff) space is normal.*

PROOF. These due to the above mentioned corollary and Dieudonné.

LEMMA 2. *If X is paracompact, then each weakly Cauchy filter with respect to such a uniformity in Lemma 1 has a cluster point.*

PROOF. Let $\mathscr{U}$ be such a uniformity for X; i. e., $\mathscr{U}$ is the family of all neighborhoods of the diagonal. Let $\mathscr{F}$ be a filter which is weakly Cauchy under $\mathscr{U}$. Let's assume that $\mathscr{F}$ has no cluster point in X. Since X is Tychonoff space, it has the Stone-Čech compactification $\beta(X)$ of X. Let A be the set of cluster points of $\mathscr{F}$ in $\beta(X)$. Then it is easily verified that Δ and $A\times X$ are disjoint closed sets in $\beta(X)\times X$, and consequently there is a neighborhood U of Δ such that $\bar{U}$ does not intersect $A\times X$, for $\beta(X)\times X$ is normal.

Since $\mathscr{U}$ is the family of all neighborhoods of the diagonal Δ, $U \in \mathscr{U}$.

Since $\mathscr{F}$ is assumed weakly Cauchy, there is a filter $\mathscr{H}_U$ stronger than $\mathscr{F}$ with $H \in \mathscr{H}_U$ and $H\times H$ contained in U. Since $\beta(X)$ is compact, $\mathscr{H}_U$ has also a cluster point in $\beta(X)$, this cluster point is also a cluster point of $\mathscr{F}$, and it clearly is not in A. This contradicts the assumption that A was the set of all cluster points of $\mathscr{F}$.

LEMMA 3. *If $(X,\mathscr{U})$ is a uniform space whose uniformity is a family of all neighborhoods of the diagonal, then the gage of $\mathscr{U}$ is the family of pseudo-metrics which are continuous on $X\times X$ with respect to the product topology.*

PROOF. Let P be the gage of $\mathscr{U}$ and P' be the family of pseudo-metrics which are continuous on $X\times X$. By definition, P is the family of pseudo-metrics which are uniformly continuous on $X\times X$. Otherhand, the uniform continuity implies the continuity. Hence $P\subset P'$. A pseudo-metric p on X is uniformly continuous on $X\times X$ relative to the product uniformity iff $V_{p,r}=\{(x,y): p(x,y)<r,\ x,\ y\in X\}$ is a member of $\mathscr{U}$ for each $r>0$. Since each member p of P' is continuous on $X\times X$ and $\mathscr{U}$ is the family of all neighborhoods of the diagonal, $V_{p,r}\in \mathscr{U}$ for each $r>0$. Therefore, p is uniformly continuous on $X\times X$. $P\supset P'$. With above result $P\subset P'$, $P=P'$

LEMMA 4. *If $(X,\mathscr{U})$ is a uniform space whose uniformity is a family of all neighborhoods of the diagonal, then the gage of $\mathscr{U}$ is the family of pseudo-metrics whose topologies are weaker than the original topology.*

PROOF. Let P be the gage of $\mathscr{U}$ and P' be the family of pseudo-metrics whose topologies are weaker than the original topology. By LEMMA 3, P is the family of pseudo-metrics which are continuous on $X\times X$. Let $p \in P'$. Then p is a continuous function on $(X,p)\times(X,p)$ and the original topology is stronger than pseudo-metric topology derived from p. Therefore, p is a continuous function on $(X, \mathscr{T}) \times (X, \mathscr{T})$, where $\mathscr{T}$ is the topology of uniformity $\mathscr{U}$. $p \in P$. $P' \subset P$. Let $p \in P$. Since P is the gage of $\mathscr{U}$, the identity map of $(X, \mathscr{U})$ onto (X, p) is uniformly continuous. The identity map of $(X, \mathscr{T})$ onto (X, p) is continuous. Hence the pseudo-metric topology derived from p is weaker than $\mathscr{T}$. $p \in P'$. $P' \supset P$. Therefore $P = P'$.

THE PROOF OF THEOREM

Proof of Necessary Condition:

Let $\mathscr{U}$ be the family of all neighborhoods of the diagonal, and let P be the family of pseudo-metrics whose topologies are weaker than the original topology. Then by LEMMA 1, $\mathscr{U}$ is the uniformity for X, and P is the gage of $\mathscr{U}$ by LEMMA 4.

By LEMMA 2, it is sufficient to prove that every filter with the given condition is weakly Cauchy filter with respect to $\mathscr{U}$.

Let $\mathscr{F}$ be any filter in X with the given condition. Let U be a member of $\mathscr{U}$. There exists the pseudo-metric $p \in P$ such that $V_{p,r} \subset U$ for some $r > 0$. Since $\mathscr{F}$ has the cluster point in (X, p), let A be the set of all cluster points of $\mathscr{F}$ in (X, p). $A \neq \phi$. For any $x \in A$, there exists the stronger filter $\mathscr{H}_x$ than $\mathscr{F}$ such that $\mathscr{H}_x$ converges to x in (X, p).

Let $\mathscr{H}_U = \sup\{\mathscr{H}_x : \mathscr{H}_x \to x \text{ in } (X, p)\}$. Then clearly $\mathscr{H}_U$ is stronger than $\mathscr{F}$ and $\mathscr{H}_U$ contains U small sets, for $\mathscr{H}_U$ converges to x, for $r > 0$, there exists a neighborhood $B_{\frac{1}{2}r}(x)$ of x such that $B_{\frac{1}{2}r}(x) \in \mathscr{H}_U$, where $B_{\frac{1}{2}r}(x) = \{y : p(x, y) < \frac{1}{2}r\}$.

Hence $B_{\frac{1}{2}r}(x) \times B_{\frac{1}{2}r}(x) \subset V_{p,r} \subset U$. $\mathscr{F}$ is weakly Cauchy filter in $(X, \mathscr{U})$. Therefore $\mathscr{F}$ has the cluster point in $(X, \mathscr{T})$.

Proof of Sufficient Condition:

Let's assume that X be not paracompact. Then there exists an open cover $\mathscr{U}$ of X which has not an open locally finite refinement.

Let $\mathscr{W}$ be the family of finite subfamilies of $\mathscr{U}$. Let's consider the family $\mathscr{F}' = \{X \sim \bigcup\{U : U \in \mathscr{U}'\} : \mathscr{U}' \in \mathscr{W}\}$. Since each member of $\mathscr{W}$ can not be a cover of X, each member of $\mathscr{F}'$ is non-void. The intersection of any two members of $\mathscr{F}'$ contains the member of $\mathscr{F}'$. Therefore, $\mathscr{F}'$ is the base for filter in X. Let $\mathscr{F}$ be the filter generated by $\mathscr{F}'$. Then $\mathscr{F}$ has no cluster point in X. For, let x be any point of X, then there exists $U \in \mathscr{U}$ such that $x \in U$. $x \notin X \sim U \in \mathscr{F}$ and $X \sim U$ is closed. $x \notin X \sim U \supseteqq \bigcap\{F : F \in \mathscr{F}\}$.

By hypothesis, there is a pseudo-metric $p \in P$ such that $\mathscr{F}$ has no cluster point in (X, p). Let's consider a family $\mathscr{V}=\{(X\sim F)^{Op}:F\in\mathscr{F}\}$ of open sets in (X, p), where A^{Op} means the interior in (X,p). Since $\mathscr{F}$ has no cluster point in (X, p), $\bigcap\{\bar{F}^{p}:F\in\mathscr{F}\}=\phi$, where $\bar{A}^{p}$ means the closure of A in (X,p). Hence $\bigcup\{X\sim\bar{F}^{p}: F\in\mathscr{F}\}= X$. That is, $\bigcup\{(X\sim F)^{Op}: F\in\mathscr{F}\}=X$.

Therefore, $\mathscr{V}$ is an open cover of (X,p). Since each open cover of a pseudo-metrizable space has the open locally finite refinement, so does $\mathscr{V}$. There exists an open locally finite refinement $\mathscr{V}'$ of $\mathscr{V}$ in (X, p).

For each $V\in\mathscr{V}'$, there exists $(X\sim F)^{Op}\in\mathscr{V}$ such that $V\subset(X\sim F)^{Op}\subsetneq X\sim F$ $=\bigcup\{U:U\in\mathscr{U}'_V\in\mathscr{W}\}$.

Otherwise, an open set in (X,p) is open in $(X,\mathscr{T})$, for the pseudo—metric topology is weaker than $\mathscr{T}$. $\{V\cap U:U\in\mathscr{U}'_V\}$ is the finite family of open sets in $(X,\mathscr{T})$.

$\mathscr{C}=\{V\cap U: U\in\mathscr{U}'_V\in\mathscr{W}, V\in\mathscr{V}'\}$ is an open locally finite refinement of $\mathscr{U}$ in $(X, \mathscr{T})$. For every $x\in X$, there exists an open neighborhood N_x of x in (X, p) such that N_x intersects only finite members $\{V_1,\cdots\cdots,V_n\}$ of $\mathscr{V}'$. Hence N_x intersects only the members of $\{U\cap V_1:U\in\mathscr{U}'_{V_1}\}\cup\cdots\cup\{U\cap V_n:U\in\mathscr{U}'_{V_n}\}$.

Therefore, N_x intersects only finite members of $\mathscr{C}$.

Otherwise, we assume that $\mathscr{U}$ has no open locally finite refinement, we arrive at the contradiction. Hence X is paracompact.

COROLLARY. *LEMMA 1. and LEMMA 2. are sufficient as well as necessary for paracompactness respectively.*

PROOF. Paracompact space $\Longrightarrow$ Lemma 1. $\Longrightarrow$ Lemma 2. $\Longrightarrow$ Theorem $\Longrightarrow$ Paracompact space.

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