CONTINUA WITH THE SAME CLASS OF HOMEOMORPHISMS

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This paper continue to study a problem proposed by S. M. Ulam that given the class of all homeomorphisms of a topological space, what other topologies exist on the same set which have these mappings as the class of all their homeomorphisms.

Let $H(X,\mathscr{U})$ be the class of all homeomorphisms of a topological space $(X, \mathscr{U})$ onto itself. It has been constructed in [1], [2], [3] and [4] many different topologies $\mathscr{V}$ for X such that $H(X,\mathscr{U})=H(X,\mathscr{V})$. However all topologies constructed for X ever since are either non-Hausdorff or non-compact. The rigid continua of DeGroot and Wille [5] which have only the identity map as homeomorphism show the existence of non-homeomorphic continua with the same class of homeomorphisms. But we are going to construct non-rigid, non-homeomorphic continua with the same class of homemorphisms by repeatedly applying the following two theorems.

First we state without proof two simple lemmas.

LEMMA 1. *Let $(X,\mathscr{U})$ be a topological space and let $P(V)$ be a topological property possessed by certain subsets V of X. If $\mathscr{V}=\{V: P(V)\}$ is a topology for X, then $H(X,\mathscr{U})\subset H(X,\mathscr{V})$.*

LEMMA 2. *Let p be a point in a Hausdorff space $(X,\mathscr{U})$. Let $P(V)$ mean that $V\in\mathscr{U}$ and $p\in V$ or $X-V$ is compact. If $\mathscr{V}=\{U:P(U)\}$, then $(X,\mathscr{V})$ is a topological space and $\mathscr{V}\subset\mathscr{U}$. Moreover, $(X,\mathscr{V})$ is a Hausdorff space if and only if $(X,\mathscr{U})$ is locally compact at all $q\neq p$.*

THEOREM 1. *Let $X,\mathscr{U},\mathscr{V}$, and p be as in Lemma 2. Suppose the following two conditions are satisfied:*

(a) *$f(p)=p$ for all f in $H(X,\mathscr{U})\cup H(X,\mathscr{V})$,*

(b) *if $p\in Cl(A)-A$ and $g\in H(X-p,U|X-p)$ then $p\in Cl(g(A))$.*

Then $H(X,\mathscr{U})=H(X,\mathscr{V})$.

PROOF. Since $f(p)=p$ for all f in $H(X,\mathscr{U}), P(V)$ is a topological property and hence by Lemma 1, $H(X,\mathscr{U})\subset H(X,\mathscr{V})$.

If $f\in H(X,\mathscr{V})$, then by (a), $f(p)=p$ for all f in $H(X,\mathscr{V})$ and by the construction of $\mathscr{V}$, we have $\mathscr{V}|X-P=\mathscr{U}|X-p$ and $f|X-p$ is bicontinuous at every q in $X-p$ relative to $U|X-$p. Since $(X,\mathscr{U})$ is Hausdorff, f is bicontinuous at each q in $X-p$ relative to $\mathscr{U}$. By (b) f and f^{-1} are also continuous at p and hence f is also in $H(X,\mathscr{U})$.

The next theorem is to reverse the order of constructing the topology, but the proof is essentially same as THEOREM 1.

THEOREM 2. *Let p be a point in a Hausdorff space $(X,\mathscr{U})$ and V in $\mathscr{U}$ which does not contain p. By $\mathscr{U}_q$ we denote the neighborhood system (not necessary open) at q in $(X,\mathscr{U})$. Let $\mathscr{V}_q=\mathscr{U}_q$ if $q\neq p$ and $\mathscr{V}_p=\{U-V;U\in\mathscr{U}_p\}$ and let $\mathscr{V}$ be the topology generated by taking $\mathscr{V}_q$ as a base of the neighborhood system at q. If the following two conditions are satisfied:*

(a) *$f(p)=p$ for all f in $H(X,\mathscr{U})\cup H(X,\mathscr{V})$,*

(b) *If $p\in Cl(A)$, then $p\in Cl(g(A))$ for each $A\subset X-p$ and $g\in H(X-p, \mathscr{U}|X-p)$.*

then $H(X,\mathscr{U})=H(X,\mathscr{V})$

PROOF. By (a) and LEMMA 1, $H(X,\mathscr{U})\subset H(X,\mathscr{V})$. If $f\in H(X,\mathscr{V})$, then by (a) again $\mathscr{U}|X-p=\mathscr{V}|X-p$, and therefore f is bicontinuous at each q in $X-p$. By (b) and (a), f and f^{-1} are also continuous at p and hence $f\in H(X,\mathscr{U})$.

In the following example, we apply THEOREM 1 and THEOREM 2 to construct different continua topologies for a set but with the same class of homeomorphisms. We show by a sequence of diagrams the spaces and procedures of construction.

In Figure 1 $(X,\mathscr{U}_1)$ is a plane continuum. Let $V=X-p$ and apply THEOREM 2 and denote the new topology by $\mathscr{U}_2$, then $H(X,\mathscr{U}_1)=H(X,\mathscr{U}_2)$ and $(X,\mathscr{U}_2)$ can be described by Figure 2 with the usual topology. By applying THEOREM 1 to $(X,\mathscr{U}_2)$ with respect to the point q and denoting the topology constructed by $\mathscr{U}_3$ we have $H(X,\mathscr{U}_2)=H(X,\mathscr{U}_3)$ and $(X,\mathscr{U}_3)$ can be described by Figure 3 with the usual topology. We then apply THEOREM 2 again to $(X,\mathscr{U}_3)$ and get $(X,\mathscr{U}_4)$ as shown in Figure 4 apply THEOREM 1 to $(X,\mathscr{U}_4)$ with respect to p and we get $(X,\mathscr{U}_5)$ as in Figure 5 which is a continuum and $H(X,\mathscr{U}_1)=H(X,\mathscr{U}_5)$.

QUESTION. Let $(X,\mathscr{U})$ be the closed interval $[0,1]$ with the usual topology. Does there exist a topology $\mathscr{V}$ for X such that $(X,\mathscr{V})$ is a continuum $\mathscr{V}(Z,\mathscr{U})$ is not homeomorphic to $(X,\mathscr{V})$ and $H(X,\mathscr{U})=H(X,\mathscr{V})$?

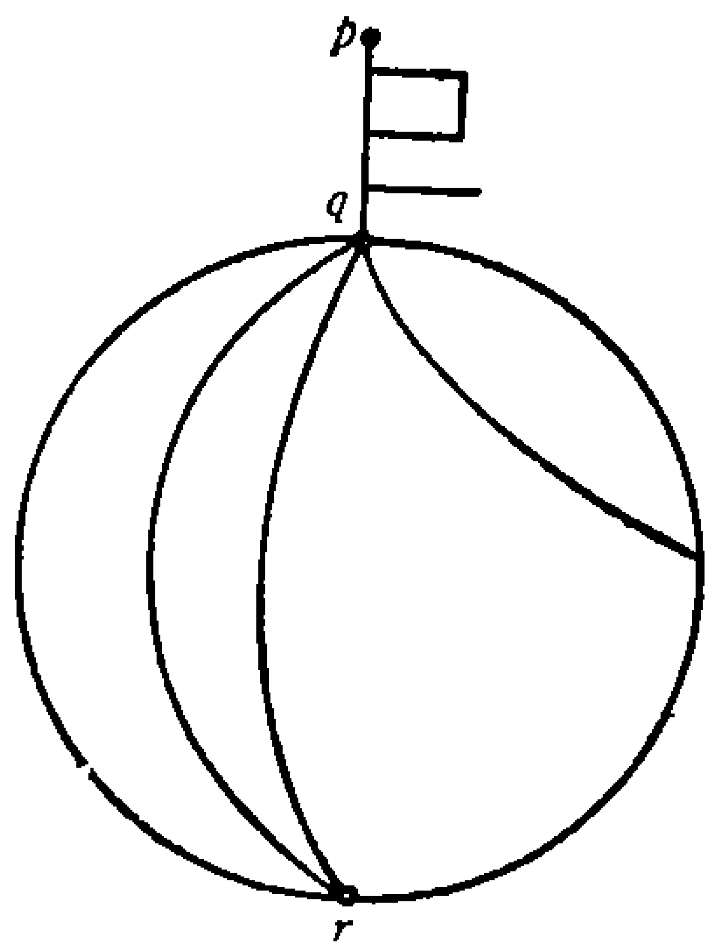

Figure 1

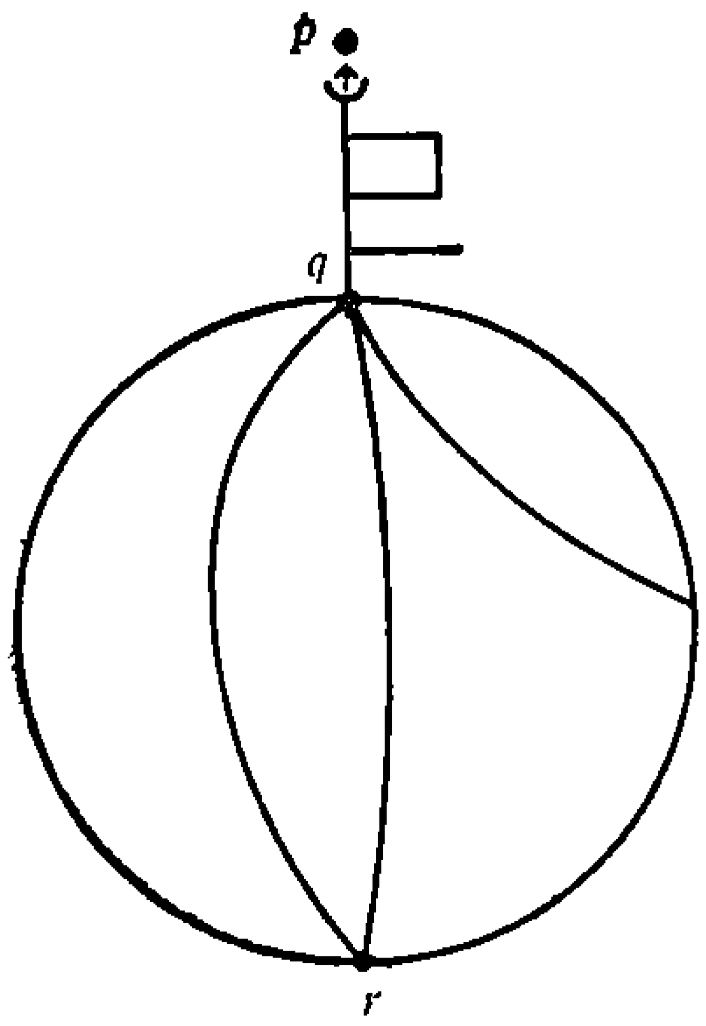

Figure 2

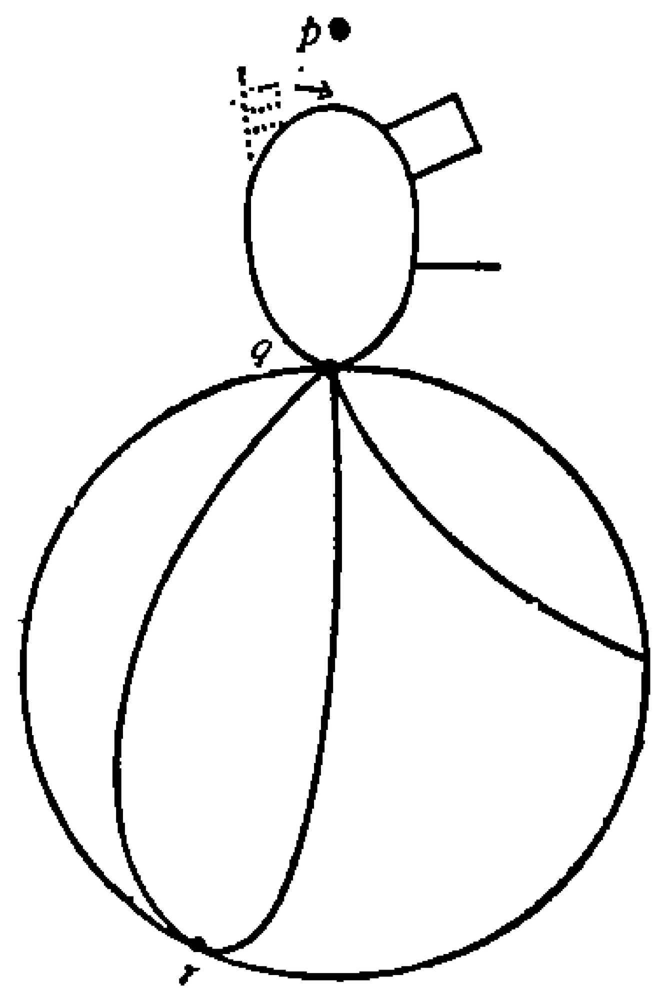

Figure 3

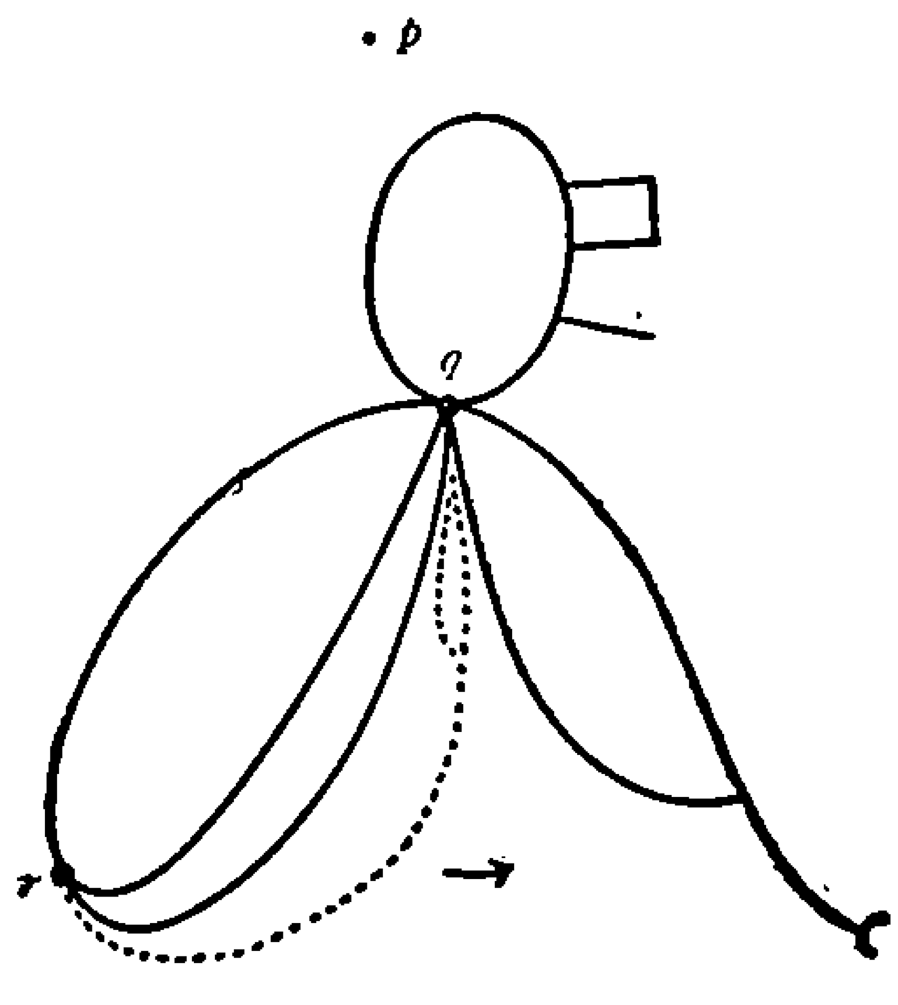

Figure 4

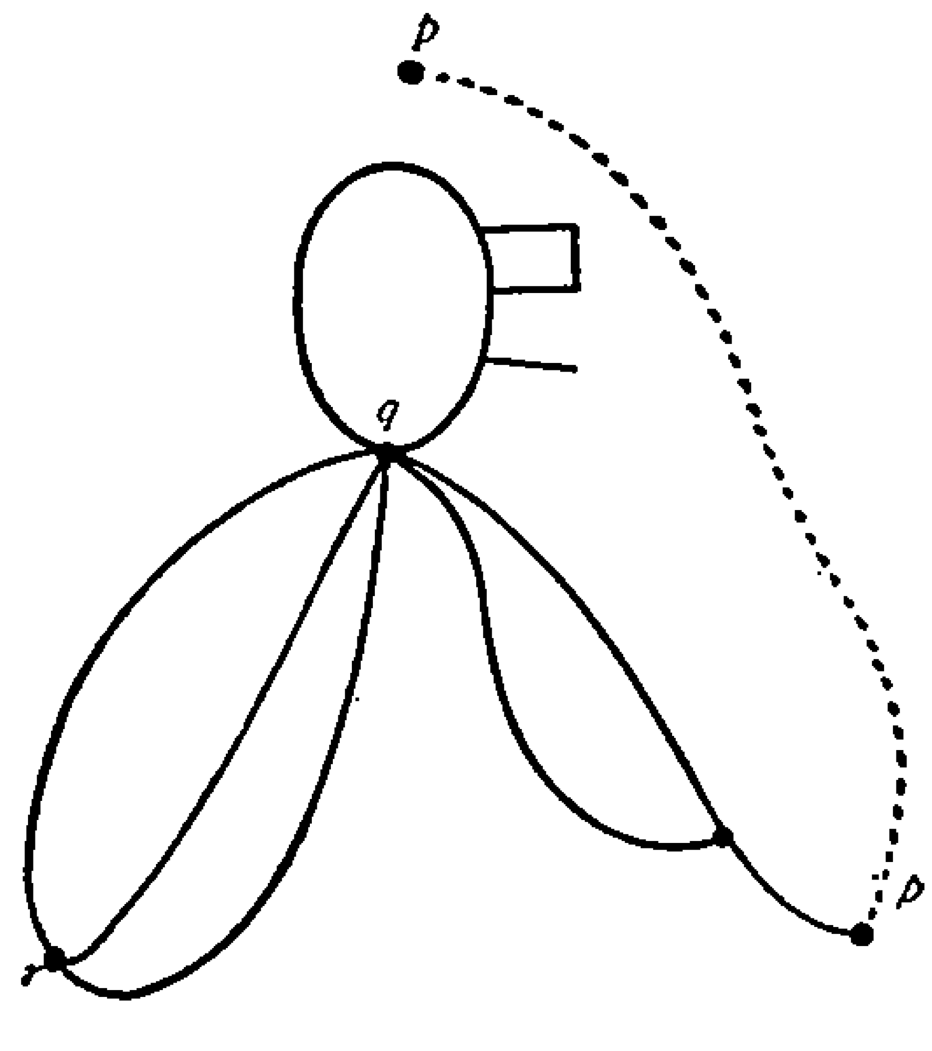

Figure 5

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