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A CHARACTERIZATION OF PARACOMPACT SPACES BY THE FILTERS IN THEM

Sung-Sa Hong

I. Introduction

The concept of a paracompact space has been introduced in 1944 by Dieudonné [1] as a generalization of certain compact spaces. In his paper, it has been proved that the product of paracompact space and every compact space (Hausdorff) is normal and that the set of all neighborhoods of the diagonal is a uniformity for it. The compactness has already been characterized through the device of the filter's formation. The purpose of this thesis is also to find a way to characterize the paracompactness in the similar filter's formation. As it has been suggested in the 2nd statement of Dieudonné's paper, the paracompactness could be formulated in terms of uniform structures. Corson proved in [1] that a T_2-space X is paracompact iff X admits a uniformity under which every filter, satisfying a Cauchy-like condition i. e. weakly Cauchy filter, has a cluster point. Based on Corson's conteribution, the present thesis attempts to construct another characterization of paracompactness by filter's formation:

A Hausdorff space $(X, \mathscr{T})$ is paracompact iff every filter in X has the cluster points in $(X, \mathscr{T})$, whenever the fliter has the cluster points in each pseudo-metric space (X, ρ) whose topology is weaker than the original topology.

In the next Preliminary section, we shall first discuss some basic concepts for the paracompactness, and these willl pave the way for the further development of the present thesis.

II. Preliminaries

In this section, we reveiew the basic concepts and terminologies relating to the paracompact space and introduce some notations that will be used in our subsequent development.

And we present the various characterizations of paracompactness which have already been found.

Definition 1. A nonvoid set X with a family $\mathscr{T}$ of subsets is called a topological space if $\mathscr{T}$ satisfies the following conditions:

a) The void set ϕ and the whole space X belong to $\mathscr{T}$.

b) The union of members of any subfamily of $\mathscr{T}$ is again a member of $\mathscr{T}$.

c) The intersection of any finite members of $\mathscr{T}$ is again a member of $\mathscr{T}$.

The family $\mathscr{T}$ is called a topology for X, and the members of $\mathscr{T}$ and called the open sets of X in this topology.

The other terminologies and theories of the topological spaces may be found either in Bourbaki [1] and [2] or Kelley [1].

Definition 2. A filter $\mathscr{F}$ in a set X is a family of non void subsets of X such that:

a) the intersecction of two members of $\mathscr{F}$ always belongs to $\mathscr{F}$; and

b) if $A \in \mathscr{F}$ and $A \subset B \subset X$, then $B \in \mathscr{F}$.

As the theory of convergence has been built on the concept of filter, we can use the concept of net instead of filter.

2.1) A filter $\mathscr{F}$ converges to a point x in a topological space X iff each neighbor hood of x is a member of $\mathscr{F}$ (i. e, the neighborhood system of x is a subfamily of $\mathscr{F}$).

2.2) In a topological space X, a point x is a culster point of filter $\mathscr{F}$ on X iff x is a closure point of each member of $\mathscr{F}$.

The other terminologies and and theories of filters including the base of filter, ultrafilter and convergence of filter, etc., are given in Bourbaki [1] and [2].

Definition 3. A metric for a set X is a function d on the cartesian product $X\times X$ to the non-negative reals such that for all points x, y and z of X,

a) $d(x, y)=d(y, x)$

b) $d(x, y)+d(y, z)\geqq d(x, z)$

c) $d(x, y)=0$ if $x=y$, and

d) if $d(x, y)=0$, then $x=y$.

A function d which satisfies only (a), (b) and (c) is called a pseudo metric on X.

From a notion of metric (pseudo metric), we can driectly derive the toplogy, i. e. metric topology (resp. pseudo metric topology) whose base is the family of all open balls:

$$\{y : d(x, y) < r ;\ r>0,\quad x, y\in X\}$$

Definition 4. A uniformity for a set X is a non void family $\mathscr{U}$ of subsets of $X\times X$ such that

a) each member of $\mathscr{U}$ contains the digonal Δ;

b) if $U\in\mathscr{U}$, than $U^{-1}\in\mathscr{U}$;

c) if $U\in\mathscr{U}$, then $V\circ V\subset U$ for some V in $\mathscr{U}$;

d) if U and V are members of $\mathscr{U}$, then $U\cap V \in\mathscr{U}$; and

e) if $U\in\mathscr{U}$ and $U\subset V\subset X\times X$, then $V\in\mathscr{U}$.

The pair $(X, \mathscr{U})$ is a uniform space.

The theory of uniform space including uniform continuity, base, subbase for uniformity, uniform topology and uniform isomorphism, etc. may be found either in Bourbaki [1], [2] or Kelley [1].

Definition 5. Afilter $\mathscr{F}$ in a uniform space $(X, \mathscr{U})$ is weakly Cauchy if for every $U\in\mathscr{U}$ some filtur stonger than $\mathscr{F}$ becomes U small. That is, there is a filter $\mathscr{H}_u$, $\mathscr{H}_u\supset\mathscr{F}$ and $H\times H\subset U$ for some $H\in\mathscr{H}_u$.

Definition 6. A family P of pseudo metrics for a set X is said to be a gage iff there is $\mathscr{U}$ for X such that P is the family of all pseudo metrics which are unifomly continuous on $X\times X$ relative to the product uniformity derived from $\mathscr{U}$, and $\mathscr{U}$ is generated by P.

Definition 7. A family $\mathscr{A}$ is a cover of a B iff B is a subset of the union $\cup\{A : A\in\mathscr{A}\}$. Expecially in topological space X, a family $\mathscr{A}$ is an open cover of X iff each member of $\mathscr{A}$ is an open set.

A subcover of $\mathscr{A}$ is a subfamily which is also a cover.

A cover $\mathscr{A}$ of a set X is a refinement of a cover $\mathscr{B}$ iff each member of $\mathscr{A}$ is a subset of a member of $\mathscr{B}$.

A cover $\mathscr{U}$ is a star-refinement of $\mathscr{V}$ iff the family of stars of $\mathscr{U}$ at points of X is a refinement of $\mathscr{V}$, where the star of $\mathscr{U}$ at $x\in X$ is the union of the members of $\mathscr{U}$ to which x belongs.

Definition 8. A family $\mathscr{A}$ of subsets of a topological space is locally finite (discrete) iff each point of the space has a neighborhood which intersects only finitely many (resp. at most one) members of $\mathscr{A}$. A family $\mathscr{A}$ is σ-locally finite (σ-discrete) iff it is the union of a countable number of locally finite (resp. discrete) subfamilies.

Definition 9. A family $\mathscr{A}$ of subsets of a topological space is closure-preserving iff, for every subfamily $\mathscr{A}'\subset\mathscr{A}$, the union of closures of members of $\mathscr{A}'$ is the closure of the union of members of $\mathscr{A}'$, and $\mathscr{A}$ is σ-closure preserving if $\mathscr{A}=\bigcup_{n=1}^{\infty}\mathscr{A}_n$, where each $\mathscr{A}_n$ is closure preserving.

If $\mathscr{U}$ and $\mathscr{V}$ are families of subsets of X, then we say that $\mathscr{V}$ is cushioned in $\mathscr{U}$ if one can assign to each $V\in\mathscr{V}$ a $U_V\in\mathscr{U}$ such that, for every $\mathscr{V}'\subset\mathscr{V}$, $\overline{\cup\{V : V\in\mathscr{V}'\}}\subset\cup\{U_V : V\in\mathscr{V}'\}$. A refinement of $\mathscr{U}$ which is cushioned in $\mathscr{U}$ is called a cushioned rgfinement of $\mathscr{U}$.

Definition 10. A topological space is fully normal iff each open cover has an open star-refinement.

Definition 11. A cover $\mathscr{U}$ of a topological space is called an even cover iff there is a neighborhood V of the diagonal in X×X such that $\{V[x] : x\in X\}$ refines $\mathscr{U}$.

This concept is derived from the Lesbesgue's covering Lemma for a pseudo metric and compact space.

Proposition 11.1) Let X be a topological space such that each open cover is even. If U is a neighborhood of the diagonal in X×X, then there is a symmetric neighborhood V of the diagonal such that $V\circ V\subset U$.

Remark. If each open cover of T_1 (regular) space, X is even, then the family of all neighborhoods of the diagonal is a uniformity for X.

Proposition 11.2) Let X be a topological space such that each open cover is even and let $\mathscr{A}$ be a locally finite (or a discrete) family of subsets of X. Then there is a neighborhood V of the diagonal in X×X such that the family of all sets V [A] for A in $\mathscr{A}$ is locally finite (resp. discrete).

The proofs of the above two propositions are given in Kelley [1].

Finally, let's define the paracompactness based on the above preparations.

Definition 12. A topological space is paracompact iff it is Hausdorff and each open cover has an open locally finite refinement.

Since it is not hard to show that a Hausdorff space is regular if each open cover has an open locally finite refinement. Hence the usual definition of paracompact space in Kelly [1] specifies regular instead of Hausdorff.

In what follows the various charcterizations of paracompactness are presented.

Theorem 1.* If X is a regular topological space, than the following statements are equivalent.

a) The space X is paracompact.

b) Each open cover of X has a locally finite refinment.

c) Each open cover of X has a colsed locally finite refinement.

d) Each open cover of X is even.

d') X is fully normal.

e) Each open cover of X has an open σ-discrete refinement.

f) Each open cover of X has an open σ-locally finite refinement.

g) Each open cover of X has a closure-preserving open refinement.

h) Each open cover of X has a closure-preserving refinement.

i) Each open cover of X has a closure-preserving closed refinement.

j) Each open cover of X has a σ-closure-preserving open refinement.

Corollary 1. A paracompact space is normal.

Corollary 2. Each pseudo metrizable space is paracompact.

Corollary 3. The family of all neighborhoods of the diagonal is the uniformity for a paracompact space.

Corollary 4. The image of a paracompact space, under a continuous, closed mapping, must be paracompact.

Theorem 2.** If X is a T_1-space, then the following statements are equivalent.

a) X is paracompact.

b) Each open cover of X has a cushioned refinement.

c) Each open cover of X has an open σ-cushioned refinement.

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* The equivalences (b), (c), (e) and (f) of Theorem 1 are due to E. Michael [1], (d) is due to J. S. Griffin and Kelley (d') to A. H. Stone [1], and (g), (h), (i), (j) to E. Michael [2].

** Theorem 2. is due to E. Michael [3]

Theorem 3. A T_2(regular) and locally compact space is paracompact iff it is the "somme" of the members of the family of the locally compact and σ-compact spaces.

This theorem is due to Bourbaki [1].

It is noted that this theorem is the relation between the paracompact space (global nature of space) and locally compact space (local nature of space).

Theorem (Dieudonné) A product space of paracompact space with a Hausdorff and compact space is also paracompact.

III. Results

Theorem 1. A Hausdorff space $(X, \mathscr{T})$ is paracompact iff every filter in X has a cluster point in $(X, \mathscr{T})$, whenever the filter has a cluster point in each pseudo metric space (X, p) whose topology is weaker than the original topology.

Lemma 1. If X is paracompact, then the set of all neighborhoods of the diagonal is a uniformity for X, and the product of X and every compact (Hausdorff) space is normal.

Proof. These follow from the Theorem 4 and Theorem 1. Corollary 3 in section II.

Lemma 2. If X is paracompact, then each weakly Cauchy filter with respect to such a uniformity in Lemma 1 has a cluster point.

Proof. Let $\mathscr{U}$ be such a uniformity for X, i.e. $\mathscr{U}$ is the family of all neighborhoods of the digonal.

Let $\mathscr{F}$ be a filter which is weakly Cauchy under $\mathscr{U}$. Let us assume that $\mathscr{F}$ has no cluster point in X. Since X is Tychonoff space, it has the Stone-Čech compactification $\beta(X)$ of X.

Let A be the set of cluster points of $\mathscr{F}$ in $\beta(X)$. Then it is easily verified that $\triangle$ and $A \times X$ are disjoint closed sets in $\beta(X) \times X$, and consequently there is a neighborhood U of $\triangle$ such that U does not intersect $A \times X$, for $\beta(X) \times X$ is normal.

Since $\mathscr{U}$ is the family of all neighborhoods of the diagonal $\triangle$, $U \in \mathscr{U}$.

Since $\mathscr{F}$ is assumed weakly Cauchy, there is a filter $\mathscr{H}_u$ stronger than $\mathscr{F}$ with $H \in \mathscr{H}_u$ and $H \times H$ contained in U. Since $\beta(X)$ is compact, $\mathscr{H}_u$ has also a cluster point in $\beta(X)$, this cluster point is also a cluster point of $\mathscr{F}$, and it clearly is not in A. This contradicts the assumption that A was the set of all cluster points of $\mathscr{F}$.

Lemma 3. If $(X, \mathscr{U})$ is a uniform space whose uniformity is a family of all neighborhoods of the diagonal, then the gage of $\mathscr{U}$ is the family of pseudo metrics which are continuos on $X \times X$ with respect to the product topology.

Proof. Let P be the gage of $\mathscr{U}$ and P' be the family of pseudo metrics which are continuos on $X \times X$. By difinition, P is the family of pseudo metrics which are uniformly continuous on $X \times X$. Otherwise the uniform continuity implies the continuity. Hence $P \subset P'$. A pseudo metric p on X is uniformly continuous on $X \times X$ relative to the product uniformity iff $V_{p,r} = \{(x, y) : p(x, y) < r,\ x, y \in X\}$ is a member of $\mathscr{U}$ for each $r > 0$. Since each member p of P' is continuous on $X \times X$ and $\mathscr{U}$ is the family of all neighborhoods of diagonal, $V_{p,r} \in \mathscr{U}$ for each $r < 0$, Therefore, p is uniformly continuous on $X \times X$. $P \supset P'$ With above result $P \subset P'$, $P = P'$.

Lemma 4. If $(X, \mathscr{U})$ is a uniform space whose uniformity is a family of all neighborhoods of the diagonal, then the gage of $\mathscr{U}$ is the family of pseudo metrics whose topologies are weaker than the original topology.

Proof. Let P be the gage of $\mathscr{U}$ and P' be the family of pseudo metrics whose topolgies are weaker then the original topology. By Lemma 3, P is the family of pseudo metrics which are continuous on $X \times X$. Let $p \in P'$. Then p is a continuous function on $(X, p) \times (X, p)$ and the original tepology is stronger than pseudo metric topology derived from p. Therefore, p is a continuous function on $(X, \mathscr{T}) \times (X, \mathscr{T})$, where $\mathscr{T}$ is the topology of uniformity $\mathscr{U}$, $p \in P$, $P' \subset P$.

Let $p \in P$. Since P is the gage of $\mathscr{U}$, the identity map of $(X, \mathscr{U})$ onto (X, p) is uniformly continuous. The identity map of $(X, \mathscr{T})$ onto (X, p) is continuous. Hence the pseudo metric topology derived from p is weaker then $\mathscr{T}$. $p \in P'$. $P' \supset P$. Therefore $P = P'$.

The proof of Theorem.

Proof of necessary condition:

Let $\mathscr{U}$ be the family of all neighborhoods of the diagonal, and let P be the family of pseudo metrics whose topologies are weaker than the original topology. Then By Lemma 1, $\mathscr{U}$ is the uniformity for X, and P is the gage of $\mathscr{U}$ by Lemma 4.

By Lemma 2, it is sufficient to prove that every filter with the given condition is weakly Cauchy filter with respect to $\mathscr{U}$.

Let $\mathscr{F}$ be any filter in X with the given condition. Let U be a member of $\mathscr{U}$. There exists the pseudo metric $p \in P$ such that $V_{p,r} \subset U$ for some $r > 0$. Since $\mathscr{F}$ has the cluster point in (X, p), let A be the set of all cluster points of $\mathscr{F}$ in (X, p). $A \neq \phi$. For any $x \in A$, there exists the stronger filter $\mathscr{H}_x$ then $\mathscr{F}$ such that $\mathscr{H}_x$ converges to x in (X, p).

Let $\mathscr{X}_U = \sup \{\mathscr{H}_x : \mathscr{H}_x \to x \text{ in } (X, p)\}$. Then clearly $\mathscr{H}_U$ is stronger than $\mathscr{F}$ and $\mathscr{H}_x$ contains U small sets, for $\mathscr{H}_U$ converges to x, for $r > 0$, there exists a neighborhood $B_{(\frac{1}{2})r}(x)$ of x such that $B_{(\frac{1}{2})r}(x) \in \mathscr{H}_U$, where $B_{(\frac{1}{2})r}(x) = \left\{y : p(x, y) < \left(\frac{1}{2}\right)r\right\}$.

Hence $B_{(\frac{1}{2})r}(x) \times B_{(\frac{1}{2})r}(x) \subset V_{p,r} \subset U$. $\mathscr{F}$ is weakly Cauchy filter in $(X, \mathscr{U})$. Therefore $\mathscr{F}$ has the cluster point in $(X, \mathscr{T})$.

Proof of sufficient condition:

Let's assume that X be not paracompact. Then there exists an open cover $\mathscr{U}$ of X which has not an open locally finite refinement.

Let $\mathscr{W}$ be the family of finite subfamilies of $\mathscr{U}$. Let's consider the family $\mathscr{F}' = \{X \sim \cup \{U : U \in \mathscr{U}'\} : \mathscr{U}' \in \mathscr{W}\}$. Since each member of $\mathscr{W}$ can not be a cover of X, each member of $\mathscr{F}'$ is nonvoid. The intersection of any two members of $\mathscr{F}'$ contains the member of $\mathscr{F}'$. Therefone, $\mathscr{F}'$ is the base for filter in X. Let $\mathscr{F}$ be the filter generted by $\mathscr{F}'$. Then $\mathscr{F}$ has no cluster point in X. For, let x be any point of X, then there exists $U \in \mathscr{U}$ such that $x \in U$. $x \overline{\in} X \sim U \in \mathscr{F}$ and $X \sim U$ is closed. $x \overline{\in} X \sim U \supseteq \bigcap_{\mathscr{F} \in \mathscr{F}} \bar{F}$

By hypothesis, there is a pseudo metric $p \in P$ such that $\mathscr{F}$ has no cluster point in (X, p). Let's consider a family $\mathscr{V} = \{\overset{op}{\widehat{X \sim F}} : F \in \mathscr{F}\}$ of open sets in (X, p), where $\overset{op}{\widehat{A}}$ means the interior in (X, p). Since $\mathscr{F}$ has no cluster point in (X, p), $\cap \{\bar{F}^p \in \mathscr{F}\} = \phi$, where $\bar{A}^p$ means the closure of A in (X, p). Hence $\cup \{X \sim \bar{F}^p : F \in \mathscr{F}\} = X$.

That is, $\cup \{\overset{op}{\widehat{X \sim F}} : F \in \mathscr{F}\} = X$.

Therefore, $\mathscr{V}$ is an open cover of (X, p). Every pseudo metrizable space is paracompact, so is (X, p). There exists an open locally finite refinement $\mathscr{V}'$ of $\mathscr{V}$ in (X, p).

For each $V \in \mathscr{V}'$, there exists $\overset{op}{\widehat{X \sim F}} \in \mathscr{V}$ such that $V \subset \overset{op}{\widehat{X \sim F}} \subseteq X \sim F = \cup \{U : U \in \mathscr{U}_V' \in \mathscr{W}\}$. Otherwise, an open set in (X, p) is open in $(X, \mathscr{T})$, for the pseudo metric topology is weaker than $\mathscr{T}$. $\{V \cap U : U \in \mathscr{U}_V'\}$ is the finite family of open sets in $(X, \mathscr{T})$.

$\mathscr{C} = \{V \cap U : U \in \mathscr{U}_V' \in \mathscr{W},\ V \in \mathscr{V}'\}$ is an open locally finite refinement of $\mathscr{U}$ in $(X, \mathscr{T})$. For every $x \in X$, there exists an open neighborhood N_x of x in (X, p) such that N_x intersects only finite members $\{V_1, \cdots\cdots, V_n\}$ of $\mathscr{V}'$. Hence N_x intersects only the members of $\{U \cap V_1 : U \in \mathscr{U}_{V_1}'\} \cup \cdots\cdots \cup \} U \cap V_n : U \in \mathscr{U}_{V_n}'\}$.

Therefore, N_x intersects only finite members of $\mathscr{C}$. Otherwise, we assume that $\mathscr{U}$ has no open locally finite refinement, we arrive at the contradiction. Hence X is paracompact.

Corollary. Lemma 1 and Lemma 2 are sufficient as well as necessary for paracompactness respectively.

Proof. Paracompact space $\Longrightarrow$ Lemma 1. $\Longrightarrow$ Lemma 2 $\Longrightarrow$ Theorem 1 $\Longrightarrow$ Paracompact space.

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(Sogang College)

<p.31에서 계속>

다. 筆者는 Universal product를 새로 導入하여 "A Note on Abelian Categories"에서 Abelian Category의 定義를 比較的 綜合的으로 定義하여 보았다.

무딘 붓을 놓으면서, Homology 代數에 趣味를 가지고 더욱 工夫하려는 讀者를 위하여 參考書籍을 들어 둔다.

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<漢陽大學校>